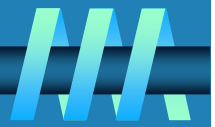
Chapter 11

Limitations of Algorithm Power

Lower Bounds

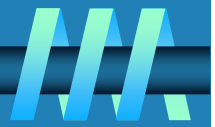


Lower bound: an estimate on a minimum amount of work needed to solve a given problem

Examples:

- number of comparisons needed to find the largest element in a set of n numbers
- number of comparisons needed to sort an array of size n
- number of comparisons necessary for searching in a sorted array
- number of multiplications needed to multiply two n-by-n matrices

Lower Bounds (cont.)

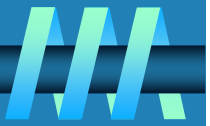


- Lower bound can be
  - an exact count
  - an efficiency class (Ω)

- *Tight* lower bound: there exists an algorithm with the same efficiency as the lower bound

Problem	Lower bound	Tightness
sorting	$\Omega(n \log n)$	yes
searching in a sorted array	$\Omega(\log n)$	yes
element uniqueness	$\Omega(n \log n)$	yes
n-digit integer multiplication	$\Omega(n)$	unknown
multiplication of n-by-n matrices	$\Omega(n^2)$	unknown

Methods for Establishing Lower Bounds



- trivial lower bounds
- information-theoretic arguments (decision trees)
- adversary arguments
- problem reduction

Trivial Lower Bounds

***Trivial lower bounds*: based on counting the number of items that must be processed in input and generated as output**

Examples

- **finding max element**
- **polynomial evaluation**
- **sorting**
- **element uniqueness**
- **Hamiltonian circuit existence**

Conclusions

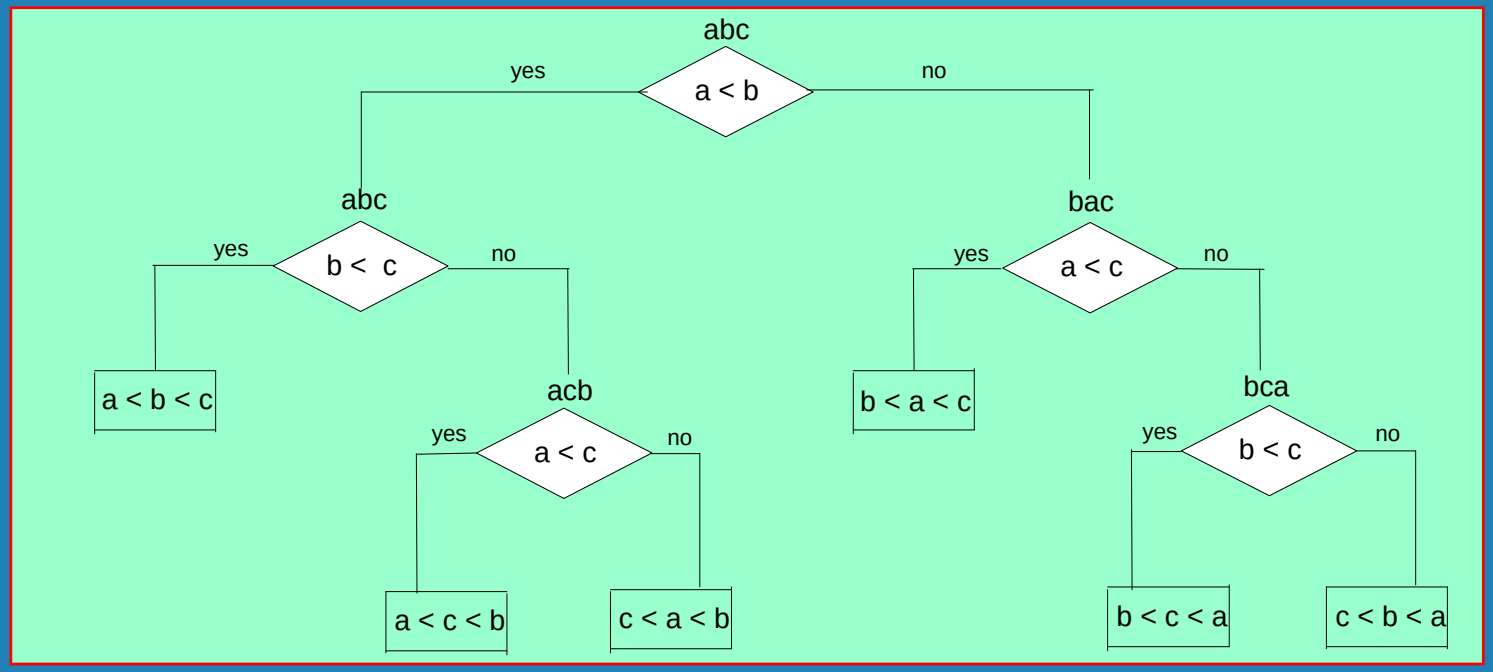
- **may and may not be useful**
- **be careful in deciding how many elements must be processed**

Decision Trees

Decision tree **— a convenient model of algorithms involving comparisons in which:**

- **internal nodes represent comparisons**
- **leaves represent outcomes**

Decision tree for 3-element insertion sort



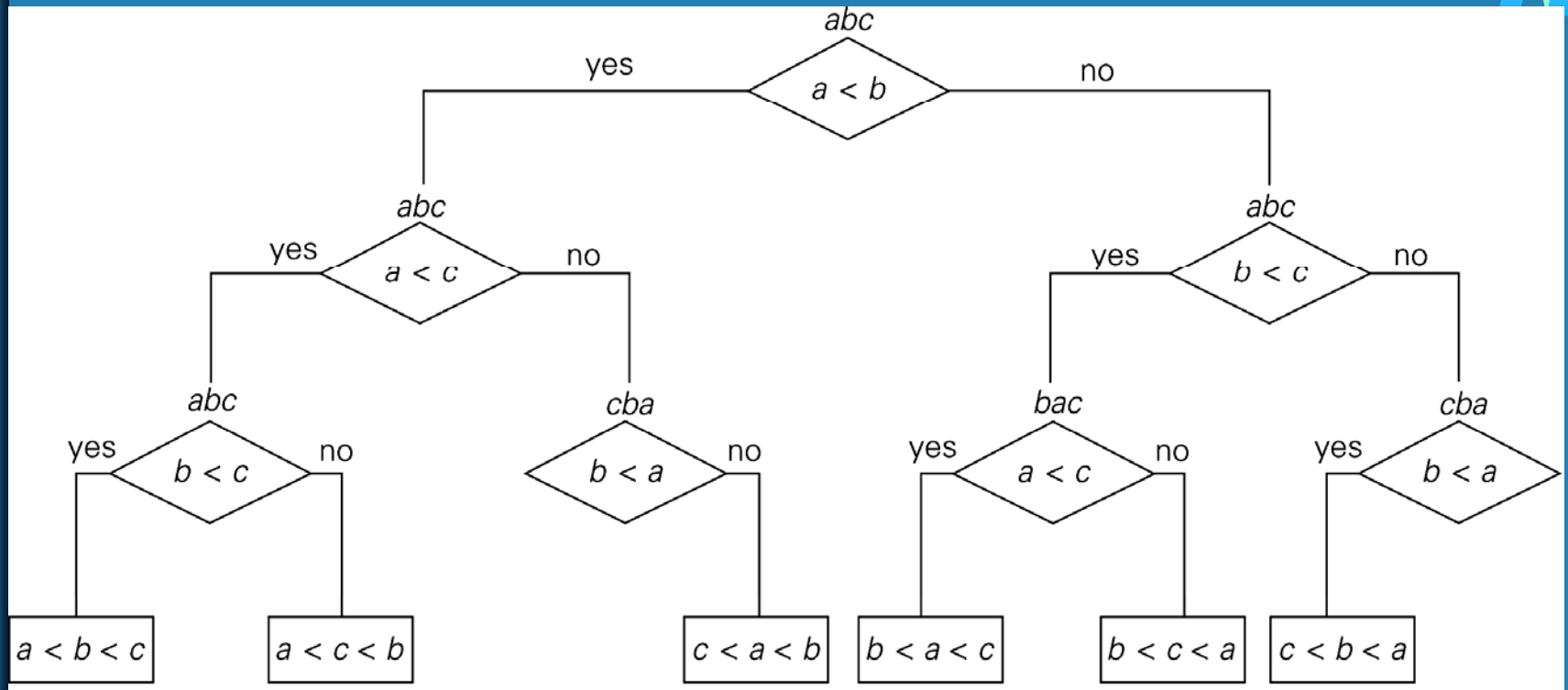


FIGURE 11.2 Decision tree for the three-element selection sort. A triple above a node indicates the state of the array being sorted. Note the two redundant comparisons $b < a$ with a single possible outcome because of the results of some previously made comparisons.

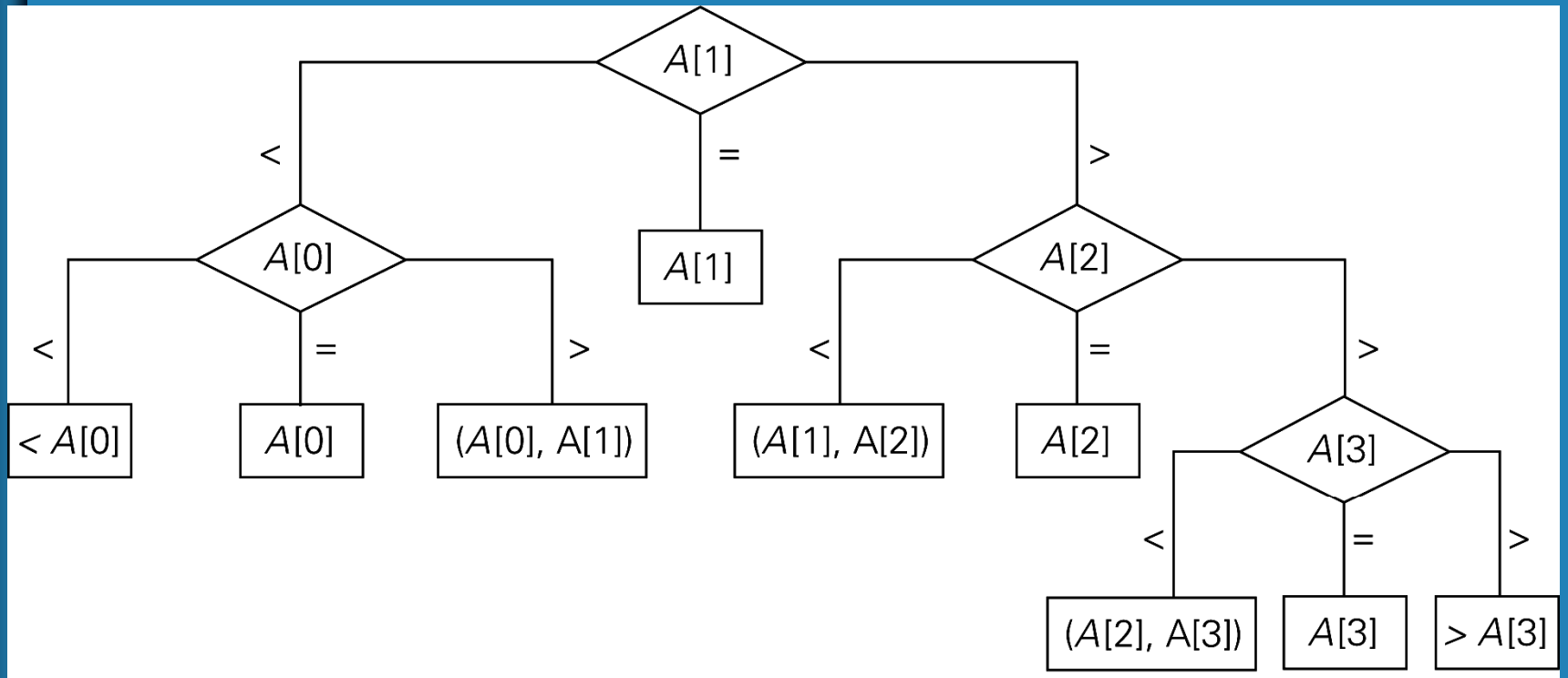


FIGURE 11.4 Ternary decision tree for binary search in a four-element array

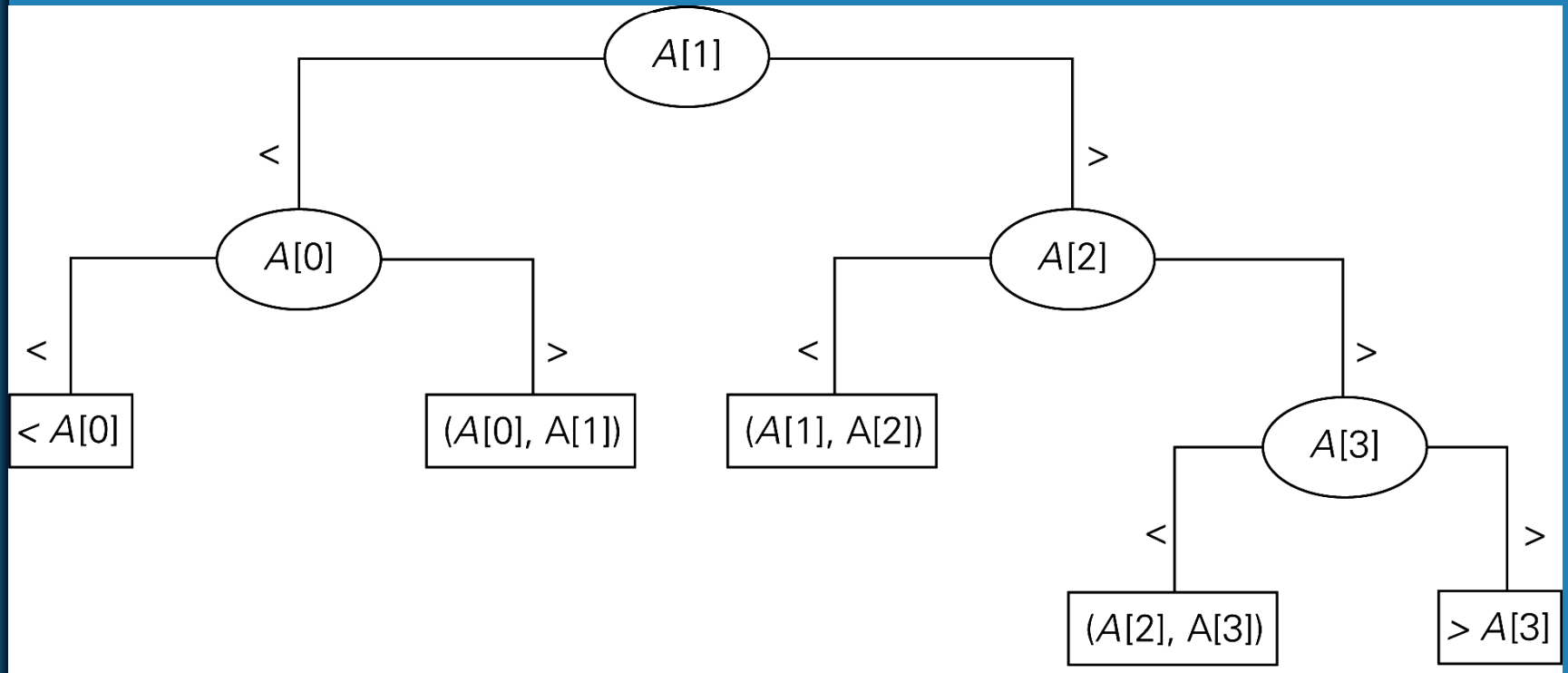
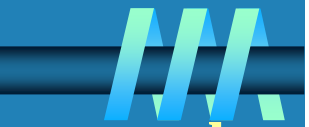


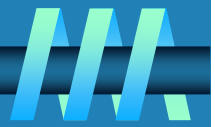
FIGURE 11.5 Binary decision tree for binary search in a four-element array

Decision Trees and Sorting Algorithms



- Any comparison-based sorting algorithm can be represented by a decision tree
- Number of leaves (outcomes) $\geq n!$
- Height of binary tree with $n!$ leaves $\geq \lceil \log_2 n! \rceil$
- Minimum number of comparisons in the worst case $\geq \lceil \log_2 n! \rceil$ for any comparison-based sorting algorithm
- $\lceil \log_2 n! \rceil \approx n \log_2 n$
- This lower bound is tight (mergesort)

Adversary Arguments



***Adversary argument*: a method of proving a lower bound by playing role of adversary that makes algorithm work the hardest by adjusting input**

Example 1: "Guessing" a number between 1 and n with yes/no questions

Adversary: Puts the number in a larger of the two subsets generated by last question

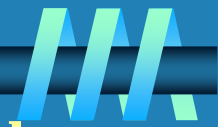
Example 2: Merging two sorted lists of size n

$a_1 < a_2 < \ldots < a_n$ and $b_1 < b_2 < \ldots < b_n$

Adversary: $a_i < b_j$ iff $i < j$

Output $b_1 < a_1 < b_2 < a_2 < \ldots < b_n < a_n$ requires $2n$-1 comparisons of adjacent elements

Lower Bounds by Problem Reduction



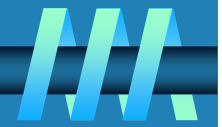
Idea: If problem *P* is at least as hard as problem *Q*, then a lower bound for *Q* is also a lower bound for *P*.

Hence, find problem *Q* with a known lower bound that can be reduced to problem *P* in question.

Example: *P* is finding MST for *n* points in Cartesian plane

Q is element uniqueness problem (known to be in $\Omega(n \log n)$)

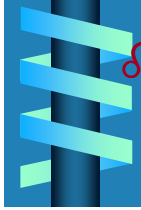
Classifying Problem Complexity



Is the problem *tractable*, i.e., is there a polynomial-time (O($p(n)$)) algorithm that solves it?

Possible answers:

- **yes (give examples)**
- **no**
  - **because it's been proved that no algorithm exists at all (e.g., Turing's *halting problem*)**
  - **because it's been be proved that any algorithm takes exponential time**
- **unknown**



Problem Types: Optimization and Decision

- *Optimization problem*: find a solution that maximizes or minimizes some objective function

- Decision problem: answer yes/no to a question

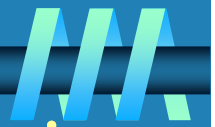
Many problems have decision and optimization versions.

E.g.: traveling salesman problem

- *optimization*: find Hamiltonian cycle of minimum length
- *decision*: find Hamiltonian cycle of length $\leq m$

Decision problems are more convenient for formal investigation of their complexity.

Class *P*

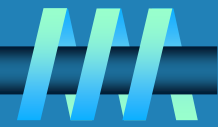


P: **the class of decision problems that are solvable in O($p(n)$) time, where $p(n)$ is a polynomial of problem's input size n**

Examples:

- **searching**
- **element uniqueness**
- **graph connectivity**
- **graph acyclicity**
- **primality testing (finally proved in 2002)**

Class *NP*



NP **(*nondeterministic polynomial*): class of decision problems whose proposed solutions can be verified in polynomial time = solvable by a *nondeterministic polynomial algorithm***

A *nondeterministic polynomial algorithm* is an abstract two-stage procedure that:

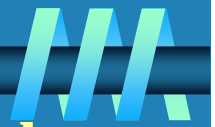
- **generates a random string purported to solve the problem**
- **checks whether this solution is correct in polynomial time**

By definition, it solves the problem if it's capable of generating and verifying a solution on one of its tries

Why this definition?

- **led to development of the rich theory called "computational complexity"**

Example: CNF satisfiability



Problem: Is a boolean expression in its conjunctive normal form (CNF) satisfiable, i.e., are there values of its variables that makes it true?

This problem is in *NP*. Nondeterministic algorithm:

- **Guess truth assignment**
- **Substitute the values into the CNF formula to see if it evaluates to true**

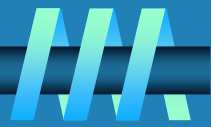
Example: (A | ¬B | ¬C) & (A | B) & (¬B | ¬D | E) & (¬D | ¬E)

Truth assignments:

A	B	C	D	E
0	0	0	0	0
		.	.	.
1	1	1	1	1

Checking phase: O(*n*)

What problems are in *NP*?



- Hamiltonian circuit existence
- Partition problem: Is it possible to partition a set of *n* integers into two disjoint subsets with the same sum?
- Decision versions of TSP, knapsack problem, graph coloring, and many other combinatorial optimization problems. (Few exceptions include: MST, shortest paths)

- All the problems in *P* can also be solved in this manner (but no guessing is necessary), so we have:

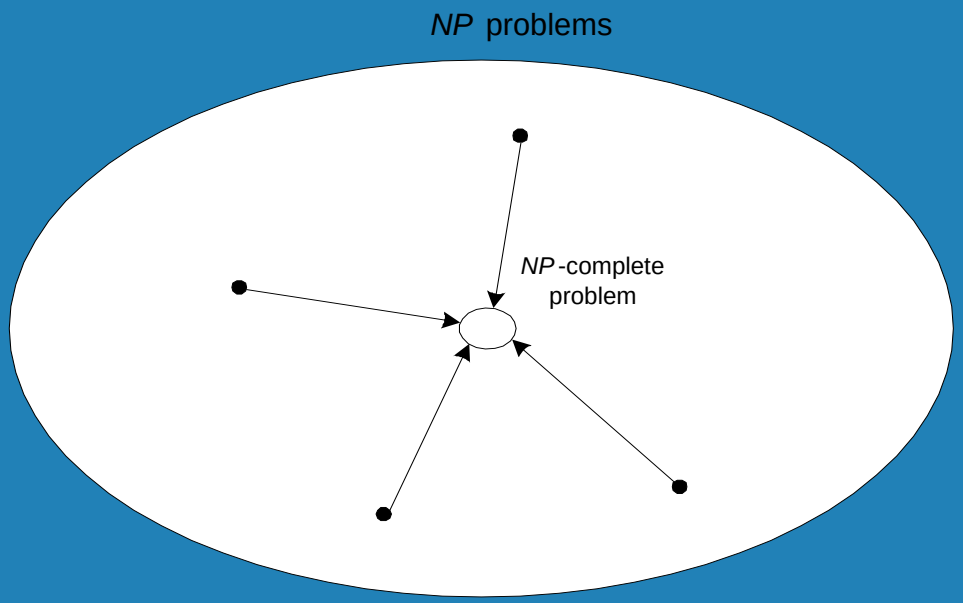
$$P \subseteq NP$$

- Big question: $P = NP$?

NP-Complete Problems

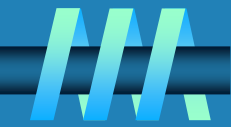
A decision problem *D* is *NP*-complete if it's as hard as any problem in *NP*, i.e.,

- ***D* is in *NP***
- **every problem in *NP* is polynomial-time reducible to *D***

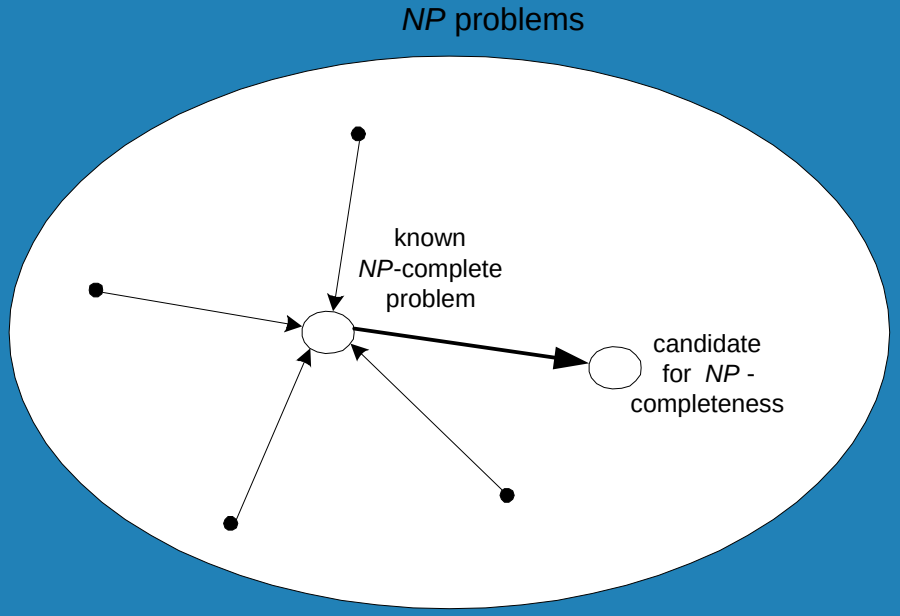


Cook's theorem (1971): CNF-sat is *NP*-complete

NP-Complete Problems (cont.)



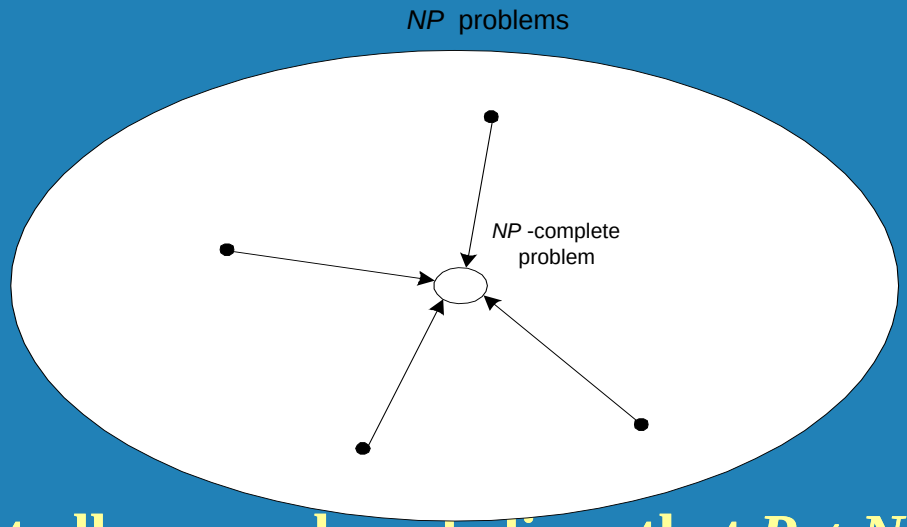
Other *NP*-complete problems obtained through polynomial-time reductions from a known *NP*-complete problem



Examples: TSP, knapsack, partition, graph-coloring and hundreds of other problems of combinatorial nature

P = *NP* ? Dilemma Revisited

- ***P*** **= *NP* would imply that every problem in *NP*, including all *NP*-complete problems, could be solved in polynomial time**
- **If a polynomial-time algorithm for just one *NP*-complete problem is discovered, then every problem in *NP* can be solved in polynomial time, i.e., *P* = *NP***



- **Most but not all researchers believe that *P* ≠ *NP* , i.e. *P* is a proper subset of *NP***

- 11.1 1, 2, 7
- 11.2 8
- 11.3 2, 9, 10