Introduction to Algorithms

Lecture 15

Last Time

- Introduction to Computational Geometry
- Computational Model
- Closest Pair problem
- Close Pair problem
- Segment intersection problem
- Orthogonal segments

Today's Topics

- Some difficult problems
- P and NP
- Polynomial-time reductions
- Cook's Theorem (SAT Problem)
- NP-complete problem
  - Clique
  - Independent set
  - Vertex cover

P vs NP
(interconnectedness of all things)

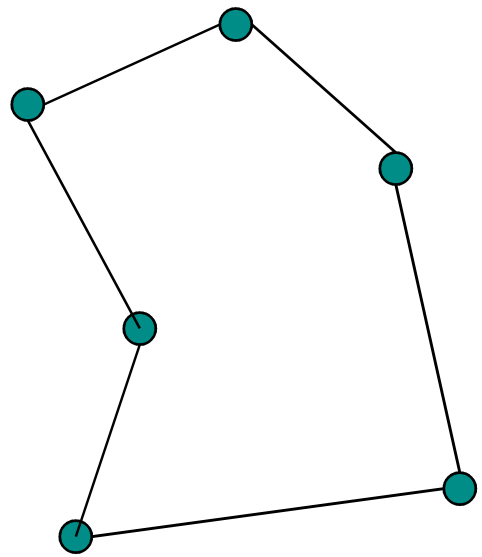
- A whole course by itself
- We'll do just one lecture
- More in "Theory of Computation", "Computational Complexity", etc.

Have seen so far

- Algorithms for various problems
  - Running times $O(nm^2)$, $O(n^2)$, $O(n \log n)$, $O(n)$, etc.
  - I.e., polynomial in the input size
- Can we solve all (or most of) interesting problems in polynomial time ?
- Not really…

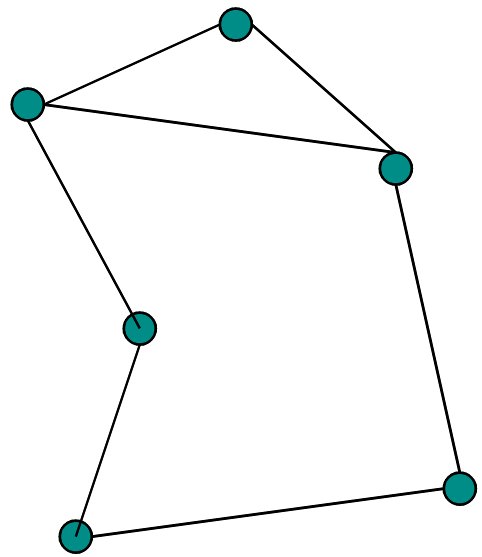
Example difficult problem

- Traveling Salesperson Problem (TSP)
  - Input: undirected graph with lengths on edges
  - Output: shortest tour that visits each vertex exactly once
- Best known algorithm: $O(n\,2^n)$ time.



Another difficult problem

- Clique:
  - Input: undirected graph $G = (V, E)$
  - Output: largest subset C of V such that every pair of vertices in C has an edge between them
- Best known algorithm: $O(n\,2^n)$ time



What can we do ?

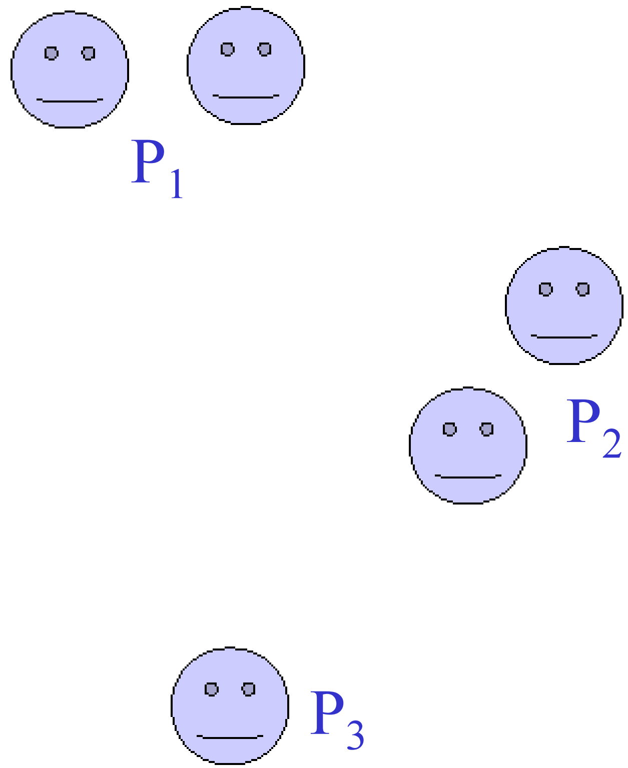
- Spend more time designing algorithms for those problems
  - People tried for a few decades, no luck
- Prove there is no polynomial time algorithm for those problems
  - Would be great
  - Seems *really* difficult
  - Best lower bounds for “natural” problems:
    - $\Omega(n^2)$ for restricted computational models
    - $4.5n$ for unrestricted computational models

What else can we do ?

- Show that those hard problems are essentially equivalent. I.e., if we can solve one of them in poly time, then all others can be solved in poly time as well.
- Works for at least 10 000 hard problems

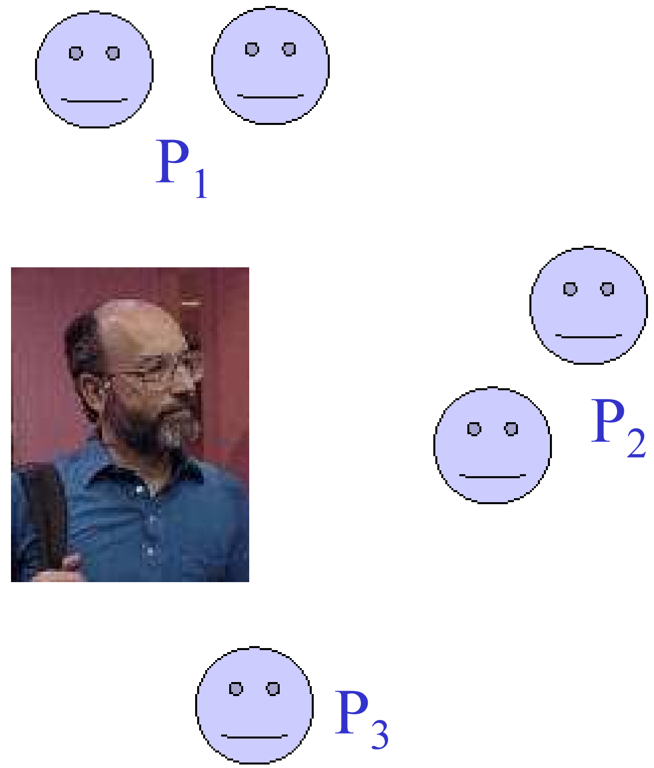
The benefits of equivalence

- Combines research efforts
- If one problem has polytime solution, then all of them do



A more realistic scenario

- Once an exponential lower bound is shown for one problem, it holds for all of them
- But someone is happy…



Summing up

- If we show that a problem Π is equivalent to ten thousand other well studied problems without efficient algorithms, then we get a very strong evidence that Π is hard.
- We need to:
  - Identify the class of problems of interest
  - Define the notion of equivalence
  - Prove the equivalence(s)

Class of problems: NP

- Decision problems: answer YES or NO. E.g.,"is there a tour of length $\leq$ K" ?
- Solvable in non-deterministic polynomial time:
  - Intuitively: the solution can be verified in polynomial time
  - E.g., if someone gives as a tour T, we can verify if T is a tour of length $\leq K$.
- Therefore, TSP is in NP.

Formal definitions of P and NP

- A problem Π is solvable in poly time (or $\Pi \in P$), if there is a poly time algorithm $V(.)$ such that for any input x:

$$\Pi(x) = \text{YES iff } V(x) = \text{YES}$$

- A problem Π is solvable in non-deterministic poly time (or $\Pi \in NP$), if there is a poly time algorithm $V(. , .)$ such that for any input x:

  $\Pi(x)$=YES iff there exists a certificate y of size poly($|x|$) such that $V(x, y)$=YES

Examples of problems in NP

- Is "Does there exist a clique in G of size $\geq K$" in NP ?

  Yes: $V(x, y)$ interprets x as a graph G, y as a set C, and checks if all vertices in C are adjacent and if $|C| \geq K$

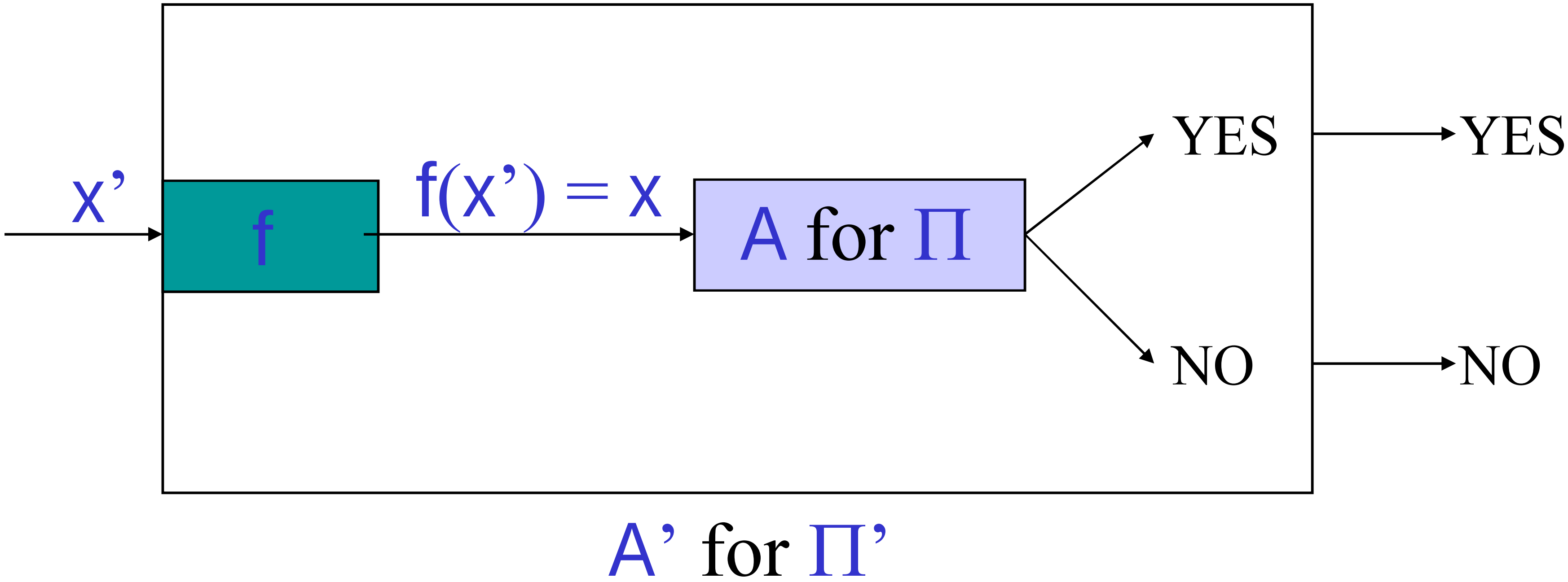
- Is Sorting in NP ?

  No, not a decision problem.

- Is "Sortedness" in NP ?

  Yes: ignore y, and check if the input x is sorted.

Reductions: Π' to Π



Reductions

- Π' is poly time reducible to Π ($\Pi' \leq \Pi$) iff there is a poly time function f that maps inputs x' of Π' into inputs x of Π, such that for any x'

$$\Pi'(x') = \Pi(f(x'))$$

- Fact 1: if $\Pi \in P$ and $\Pi' \leq \Pi$ then $\Pi' \in P$
- Fact 2: if $\Pi \in NP$ and $\Pi' \leq \Pi$ then $\Pi' \in NP$
- Fact 3: if $\Pi' \leq \Pi$ and $\Pi'' \leq \Pi'$ then $\Pi'' \leq \Pi$

Showing equivalence between difficult problems

- Options:
  - Show reductions between all pairs of problems
  - Reduce the number of reductions (!) using transitivity of "$\leq$"
  - Show that *all* problems in NP a reducible to a *fixed* Π. To show that some problem $\Pi' \in NP$ is equivalent to all difficult problems, we only show $\Pi \leq \Pi'$.