Introduction to Algorithms

Lecture 11

Last Time

- Optimization Problems
- Greedy Algorithms
- Graph Representation & Algorithms
- Minimum Spanning Tree
  - Prim's Algorithm
  - Kruskal's Algorithm

Today's Topics

- Shortest paths
- Dijkstra's algorithm
- Breadth-First-Search
- Depth-First-Search
- Connected Components

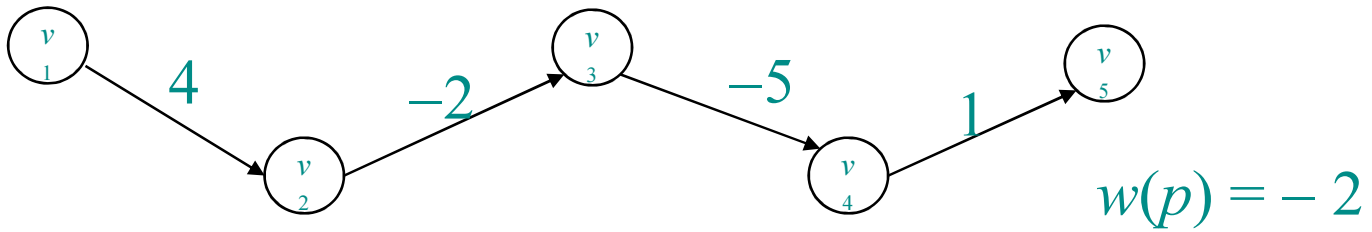
Shortest Paths in Weighted Graphs

Digraph $G = (V, E)$ with weight function $w: E \to R$

Weight of path $p = v_1 \to v_2 \to \ldots \to v_k$ is

$$w(p) = \sum_{i=1}^{k-1} w(v_i, v_{i+1})$$

E.g.,

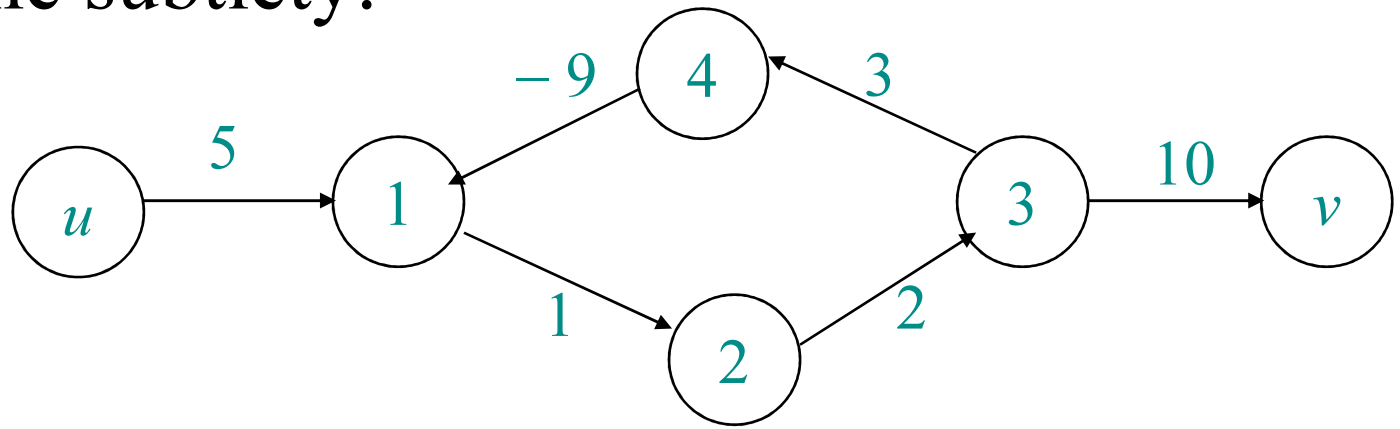


Shortest Paths in Weighted Graphs

Shortest path from u to v is a path of minimum weight from u to v. The shortest path weight is the weight of such a path:

$$\delta(u,v) = \begin{cases} \min\{w(p) : u \xrightarrow{p} v\} & \text{If there is a path from } u \text{ to } v, \\ \infty & \text{Otherwise.} \end{cases}$$

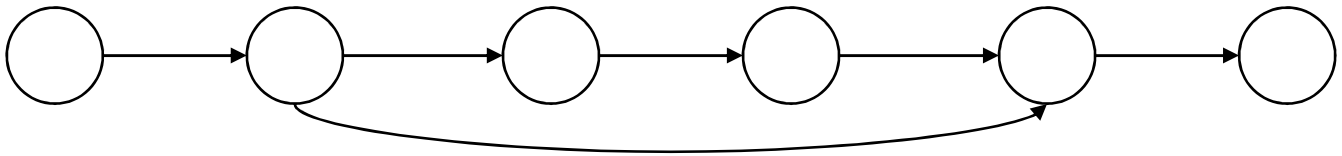
One subtlety:



Optimal Substructure

Theorem: Subpaths of shortest paths are shortest paths.

Proof: Cut and paste:



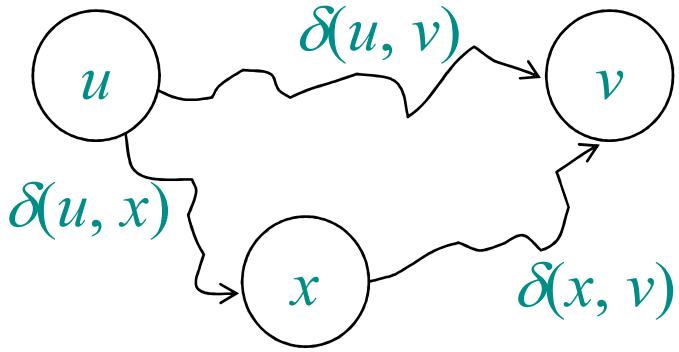
If some subpath were NOT a shortest path, could substitute the shorter subpath and create a shorter total path.

Triangle Inequality

Theorem: For all $u, v, x \in V$,

$$\delta(u, v) \leq \delta(u, x) + \delta(x, v).$$

Proof:



If triangle inequality violates then $u \rightsquigarrow x \rightsquigarrow v$ is a shorter path than $u \rightsquigarrow v$, contradicting statement that $\delta(u, v)$ corresponds to a shortest path.

Single-source shortest paths problem

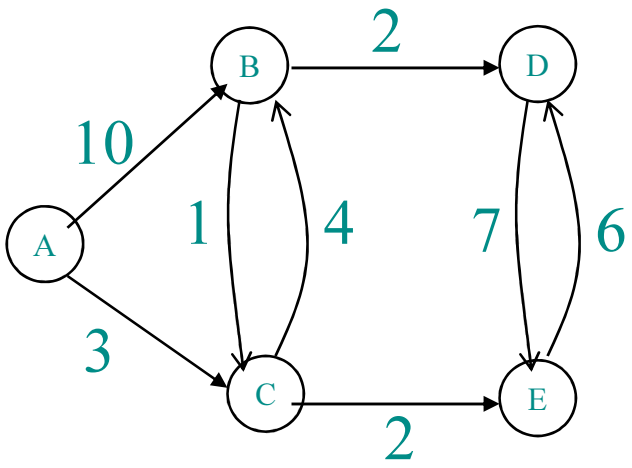
Goal: From a given source vertex $s \in V$, find the shortest-path weight $\delta(s, v)$ for all $v \in V$.

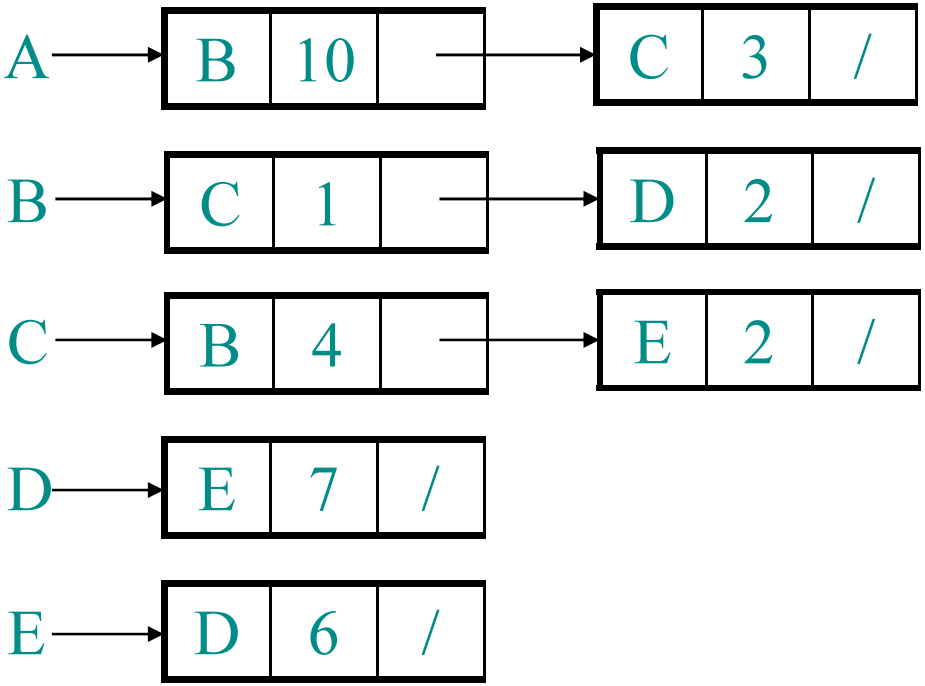
Here we assume $w(u, v) \geq 0$, so $\delta(s, v) \geq -\infty$.

Some variants:

- Single-destination shortest-path problem
- Single-pair shortest-path problem
- All-pairs shortest-path problem

Adjacency-List Representation





Mini-Priority Queue

A data structure for maintaining a set S of elements, each with an associated value (key), supporting:

- Insert(S, x): insert the element x into S.
- Minimum(S): returns element with smallest key.
- Extract-Min(S): return and removes element with smallest key.
- Decrease-Key(S, x, k): decreases the value of element x's key to k.

Dijkstra's Algorithm

Idea: Greedy Algorithm

(only valid for non-negative weights)

1. Maintain set S of vertices whose shortest path distance from s are known.
2. At each step, add to S the vertex $u \in V - S$ whose distance estimate from s is minimum.
3. Update distance estimates of vertices adjacent to u.

Dijkstra(G, w, s)

$d[s] \leftarrow 0$

$d[v] \leftarrow \infty$ for each $v \in V - \{s\}$

$S \leftarrow \varnothing$

$Q \leftarrow V$ (priority queue of vertices keyed by d)

while $Q \neq \varnothing$

do $u \leftarrow$ Extract-Min(Q)

$S \leftarrow S \cup \{u\}$

for each $v \in$ Adj[u]

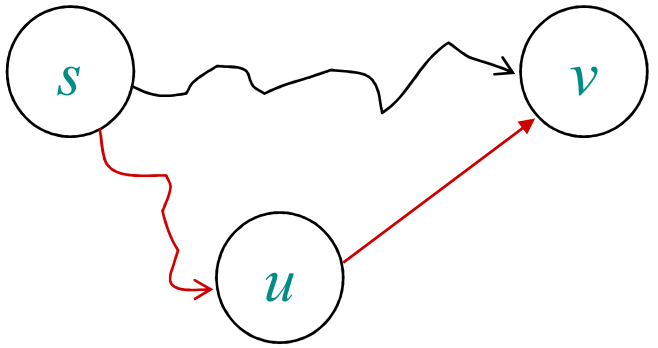
do if $d[v] > d[u] + w(u, v)$

then $d[v] \leftarrow d[u] + w(u, v)$

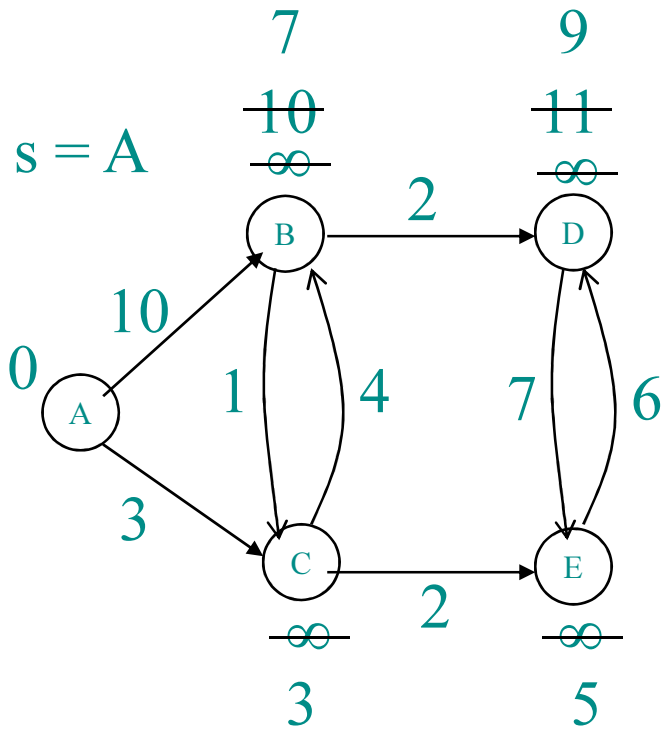
relaxation step

Implicit Decrease-Key

Maintain set of distance estimates, d and update to shortest-path weights before adding to S



Example



Q:	A	B	C	D	E
d:	0	∞	∞	∞	∞
		10	3	∞	∞
		7		∞	5
		7		11	
				9	

S = {A, C, E, B, D}

Correctness (Part I)

Lemma: Invariant: $d[v] \geq \delta(s, v)\ \forall v \in V$ at all times.

Proof:

Init $d[s] = 0$ and $d[v] = +\infty$ for $v \neq s$; $\delta(s, s) = 0$ and $\delta(s, v) \leq \infty\ \forall v$, so OK.

Suppose invariant fails, that v is the first vertex with $d[v] < \delta(s, v)$ and u is the vertex that cause $d[v]$ to change by $d[v] = d[u] + w(u,v)$.

Correctness (Part I)

Then

$d[v] < \delta(s, v)$ supposition

$\leq \delta(s, u) + \delta(u, v)$ triangle inequality

$\leq \delta(s, u) + w(u, v)$ shortest path $\leq$ specific path

$\leq d[u] + w(u, v)$ v is first violation, so $\delta(s, u) \leq d[u]$

$d[v] < d[u] + w(u, v)$ contradiction!

Correctness (Part II)

Theorem: When Dijkstra's algorithm terminates,

$$d[v] = \delta(s, v) \; \forall v \in V.$$

Proof: $d[v]$ doesn't change once added to S, so suffices to show true when added.

Suppose u is the first vertex about to be added to S for which $d[u] \neq \delta(s, u)$.

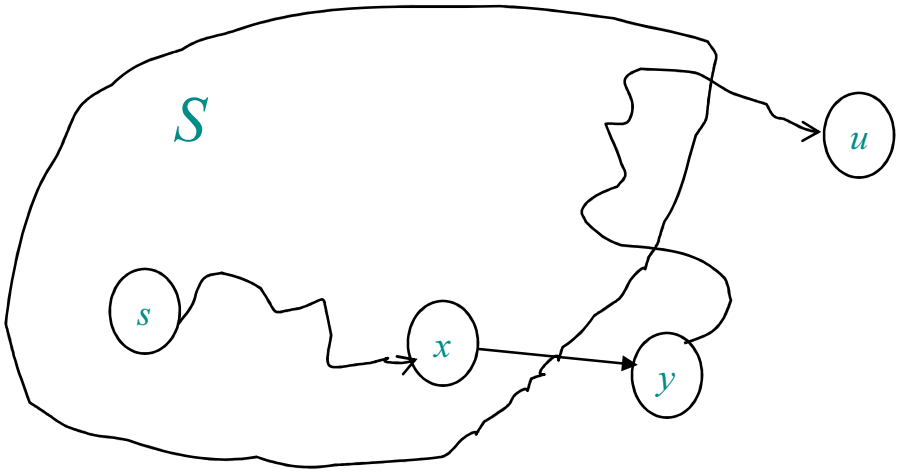
$\Rightarrow d[u] > \delta(s, u)$ by previous lemma.

Let p be a shortest path from s to u [$w(p) = \delta(s, u)$]

Consider first place p exits S [via edge (x, y)]

(y is first vertex along p in $V - S$; x is predecessor of y along p)

Correctness (Part II)



Because u is first violation, $d[x] = \delta(s, x)$.

When x was added to S, we relaxed (x, y) and set

$d[y] = \delta(s, x) + w(x, y) = \delta(s, y)$ because subpaths of shortest path are shortest paths

Thus $d[y] = \delta(s, y) \leq \delta(s, u) \leq d(u)$

sub-path previous lemma

Correctness (Part II)

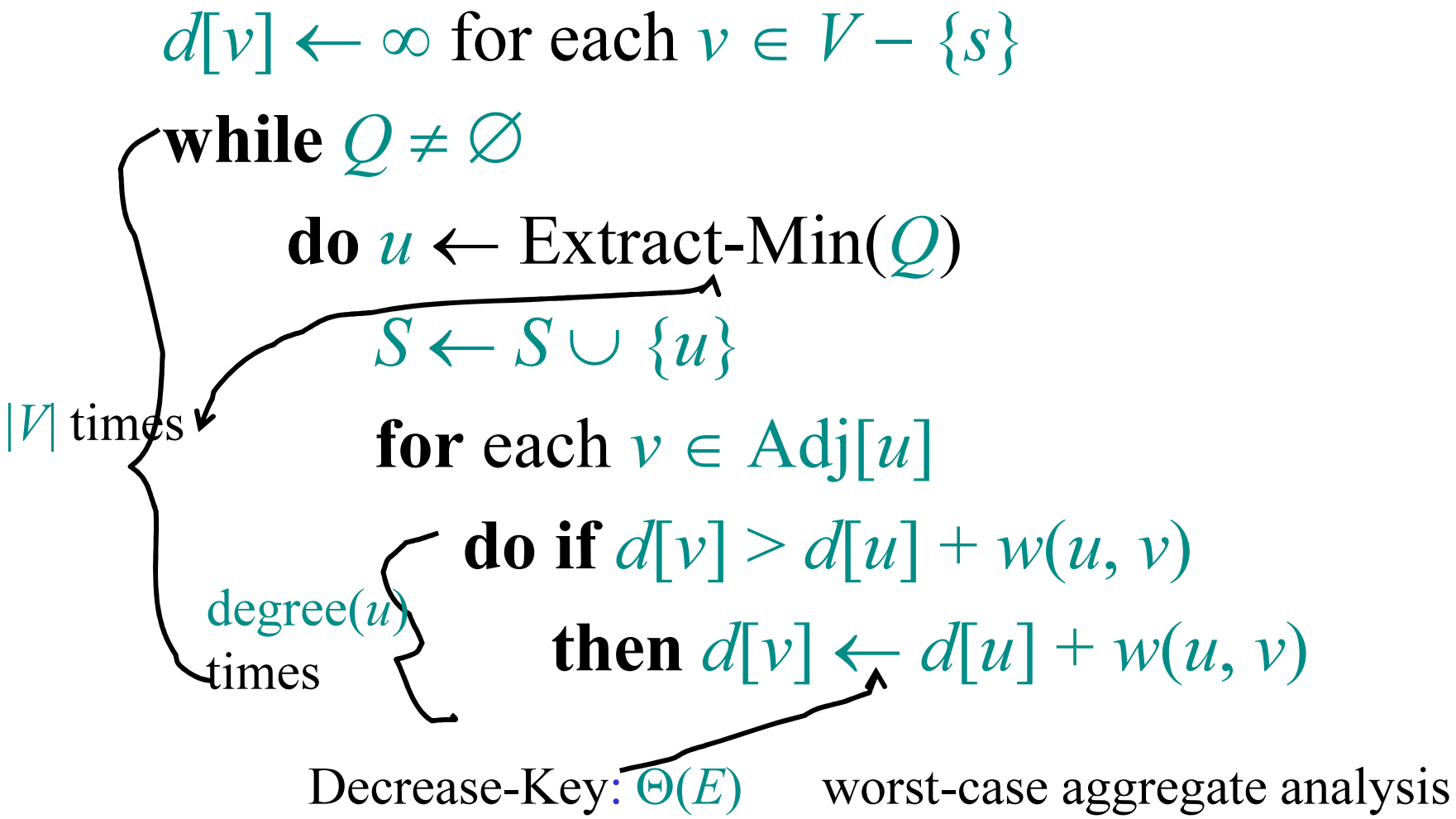
But $d[u] \le d[y]$ by Dijkstra's choice of u

Emphasizes need for greedy step.

So $d[y] = \delta(s, y) = \delta(s, u) = d[u]$.

Contradiction!

Analysis


$d[v] \leftarrow \infty$ for each $v \in V - \{s\}$
while $Q \neq \varnothing$
do $u \leftarrow$ Extract-Min(Q)
$S \leftarrow S \cup \{u\}$
$|V|$ times
for each $v \in$ Adj$[u]$
do if $d[v] > d[u] + w(u, v)$
degree(u) times
then $d[v] \leftarrow d[u] + w(u, v)$
Decrease-Key: $\Theta(E)$
worst-case aggregate analysis

Analysis

$$\text{Time} = \Theta(V) \cdot T_{\text{Extract-Min}} + \Theta(E) \cdot T_{\text{Decrease-Key}}$$

(same as Prim's MST algorithm)

Q	$T_{\text{Extract-Min}}$	$T_{\text{Decrease-Key}}$	Total
array	$\Theta(V)$	$\Theta(1)$	$\Theta(V^2)$
binary heap	$\Theta(\lg V)$	$\Theta(\lg V)$	$\Theta(E \lg V)$
Fibonocci heap	$\Theta(\lg V)$	$\Theta(1)$	$\Theta(E + V \lg V)$
	amortized	amortized	worst case

Unweighted Graphs

Suppose $w(u, v) = 1 \ \forall (u, v) \in E$. Then Dijkstra's algorithm can be improved using FIFO queue in place of priority queue.

- BFS
- DFS

Google Problem: Graph Searching

- How to search and explore in the Web Space?
- How to find all web-pages and all the hyperlinks?
  - Start from one vertex "www.yahoo.com"
  - Systematically follow hyperlinks from the discovered vertex/web page
  - Build a search tree along the exploration

Breadth-First-Search

BFS(G, w, s) Time: $\Theta(V + E)$

$d[s] \leftarrow 0$

$d[v] \leftarrow \infty$ for each $v \in V - \{s\}$

$Q \leftarrow \{s\}$

while $Q \neq \varnothing$

do $u \leftarrow$ Dequeue(Q)

for each $v \leftarrow$ Adj[u]

do if $d[v] = \infty$

then $d[v] \leftarrow d[u] + 1$

Enqueue(Q, v)

Correctness of BFS

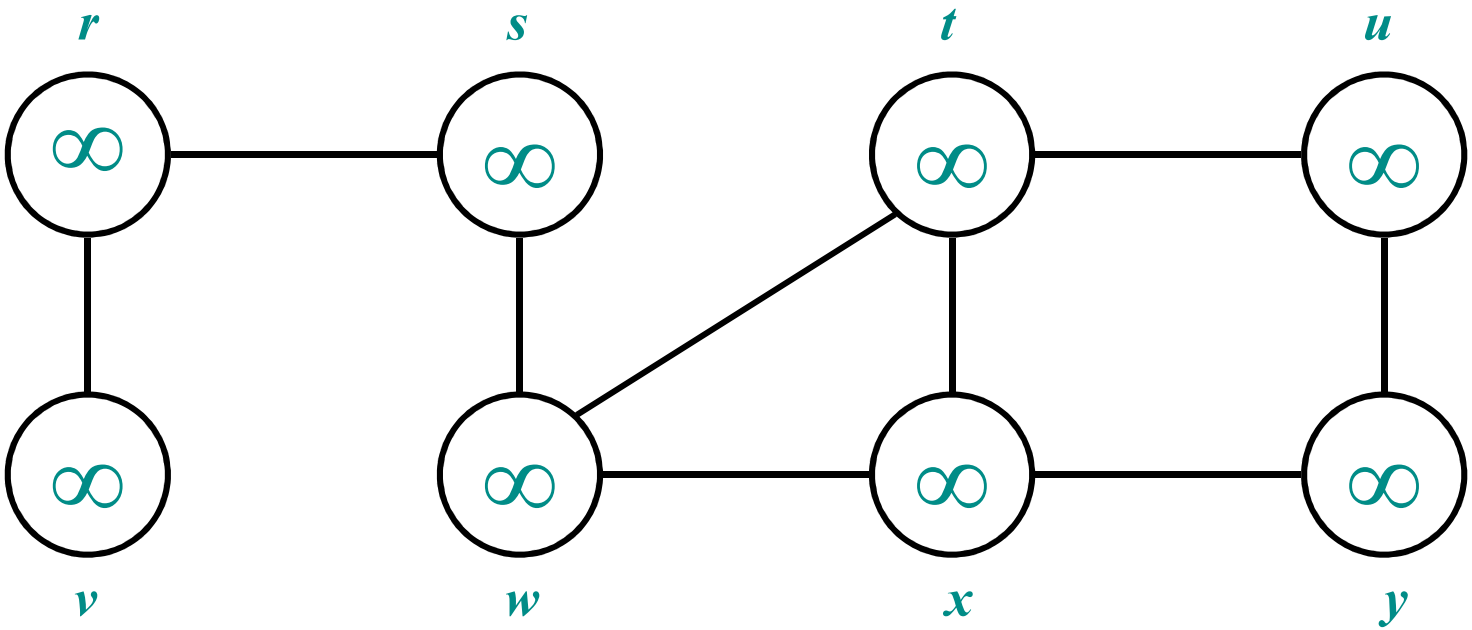
Key Idea: FIFO queue in BFS mimics priority queue in Dijkstra.

Invariant: v immediately after u in queue

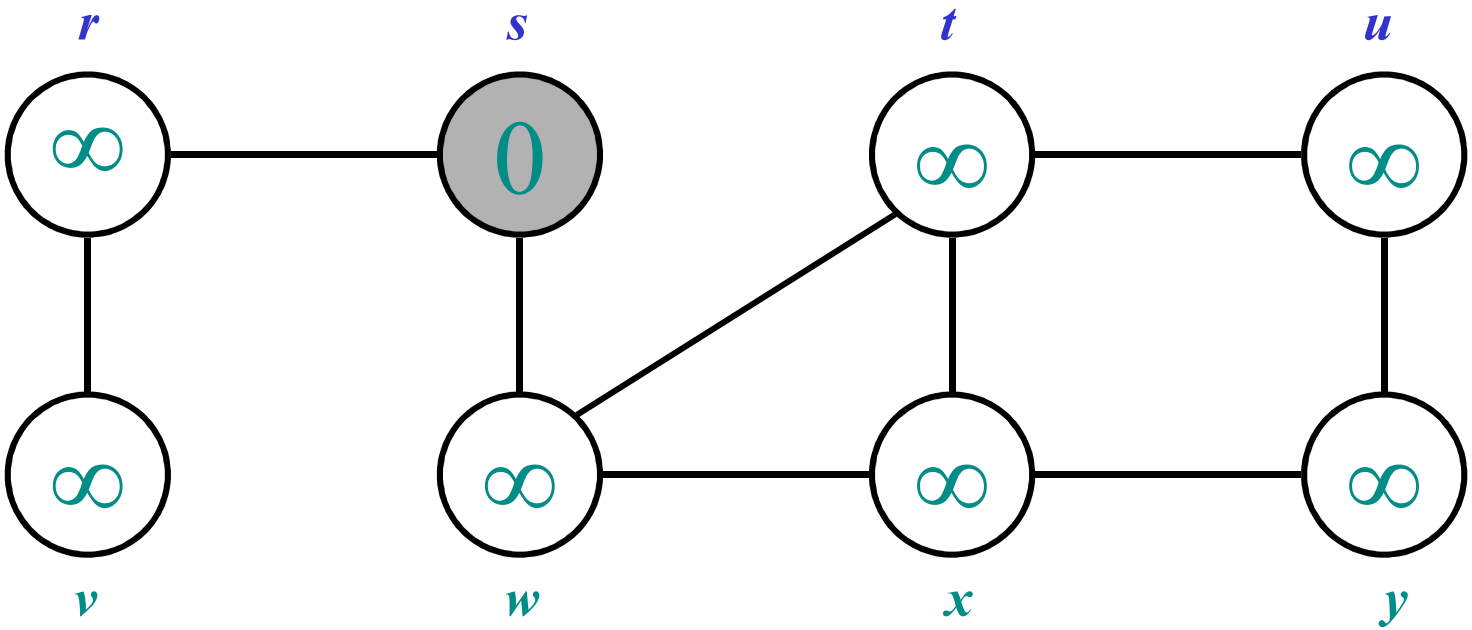
$\Rightarrow d[v]$ is either $d[u]$ or $d[u] + 1$.

(At all times, there are two distinct d values in the queue.)

Example

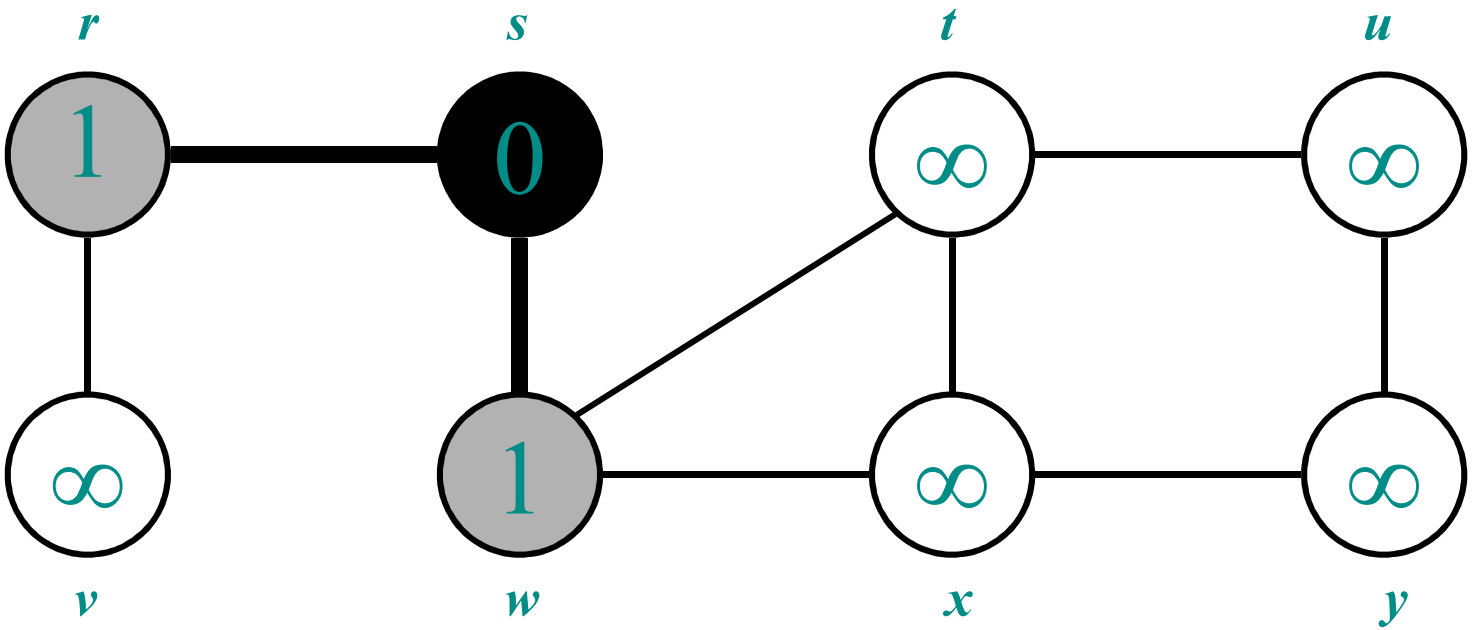


Example



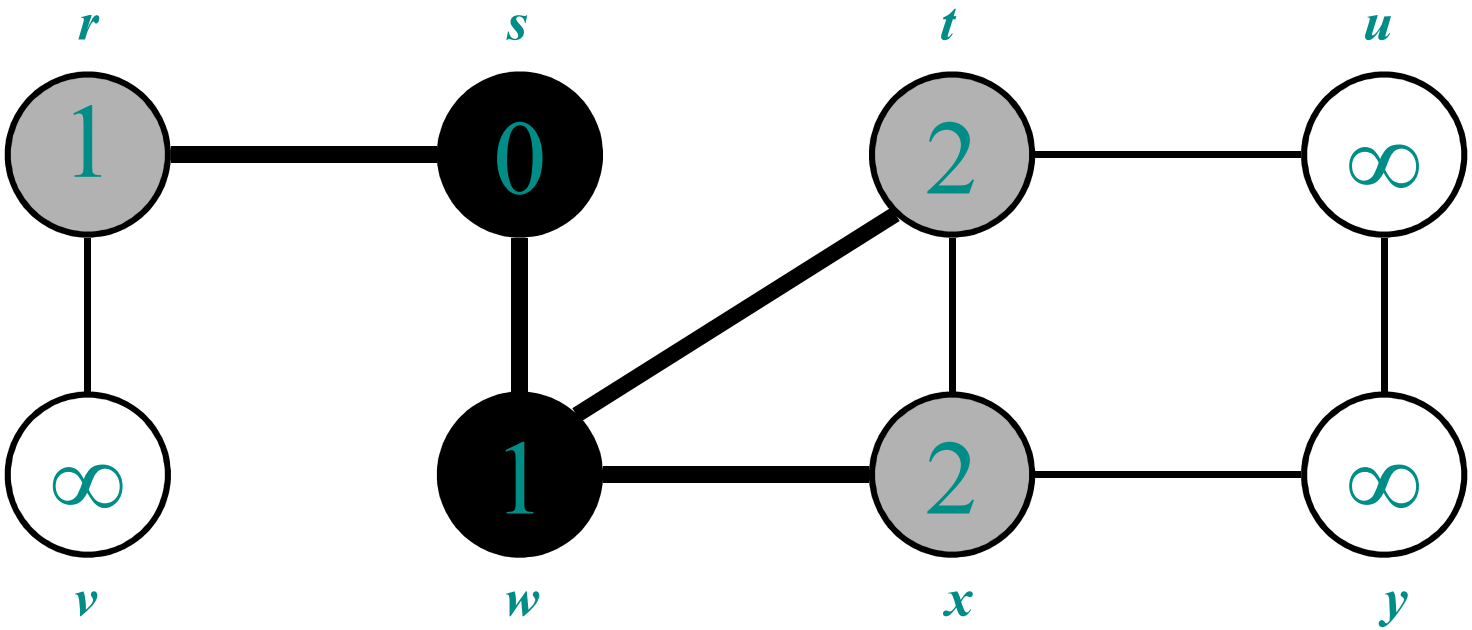
Q: | *s* |

Example

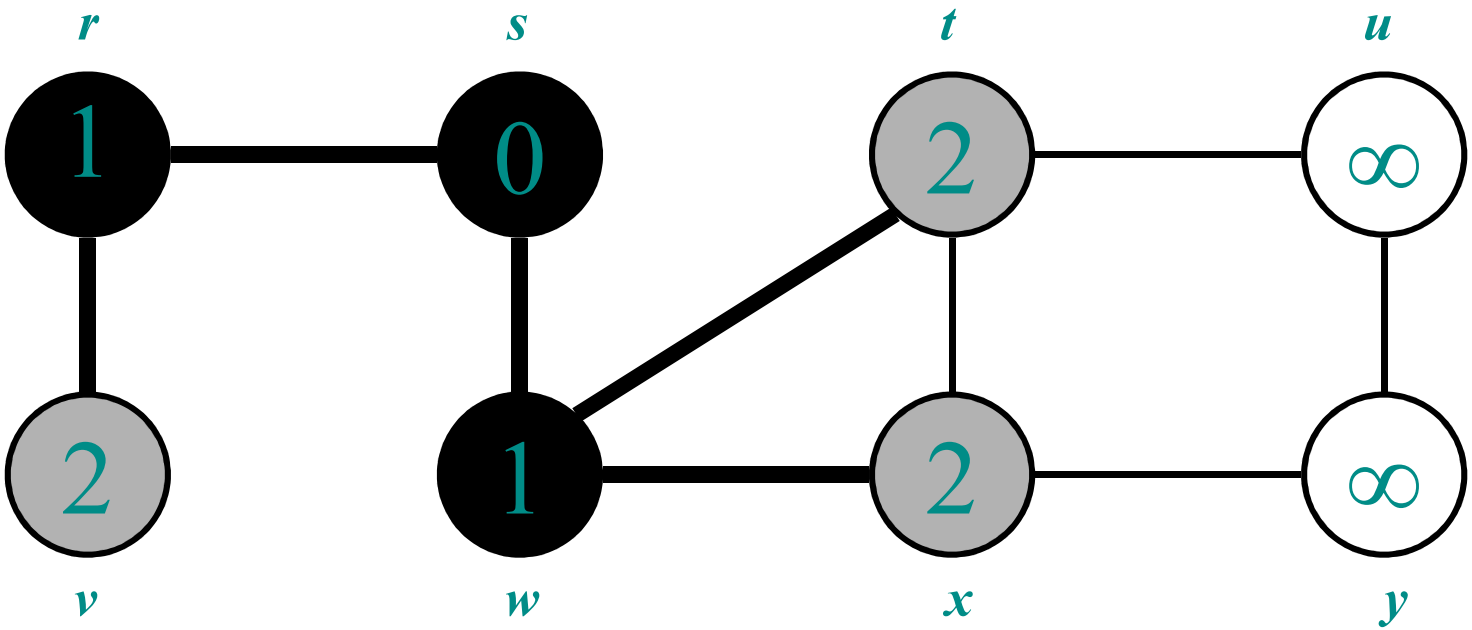


Q: | *w* | *r* |

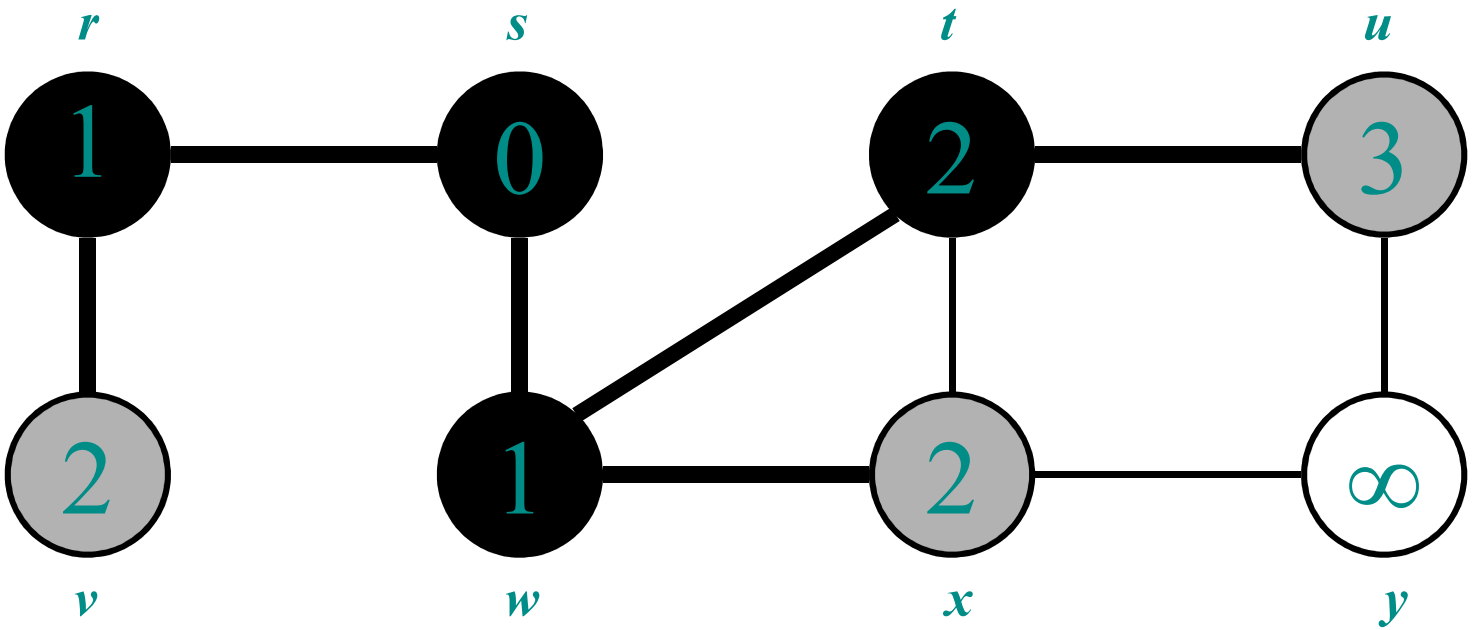
Example



Example

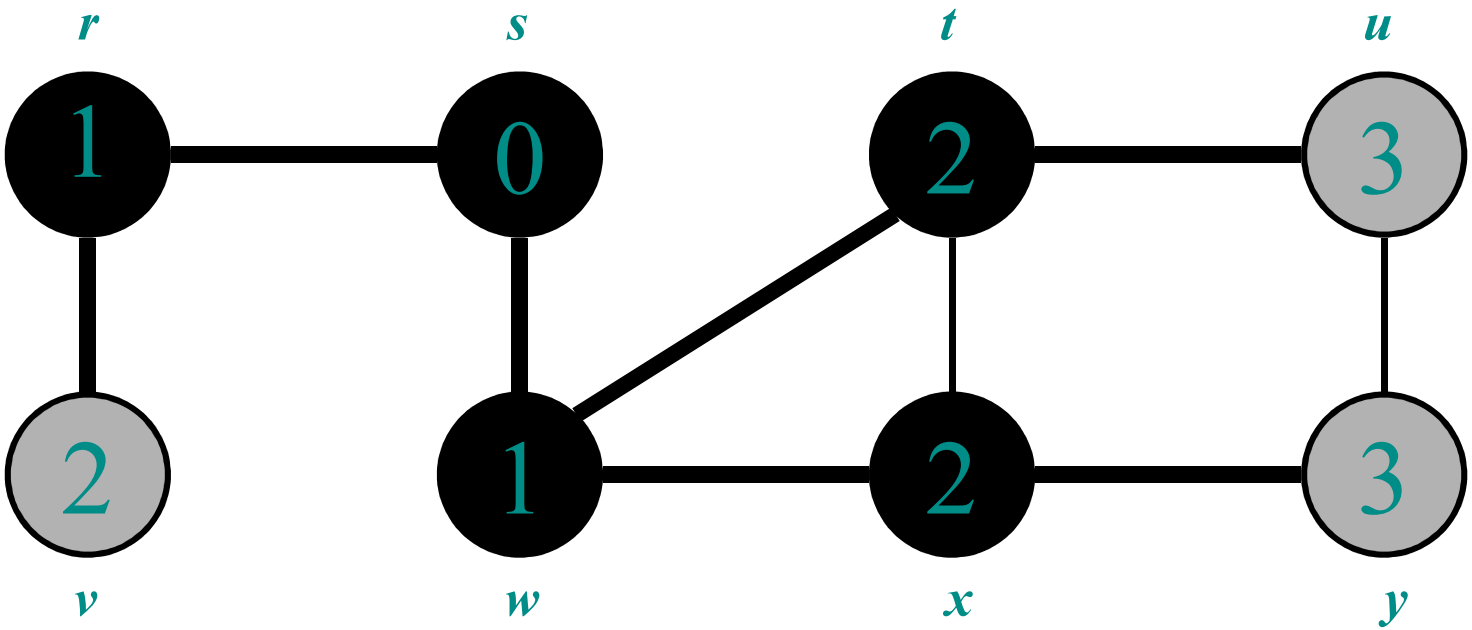


Example



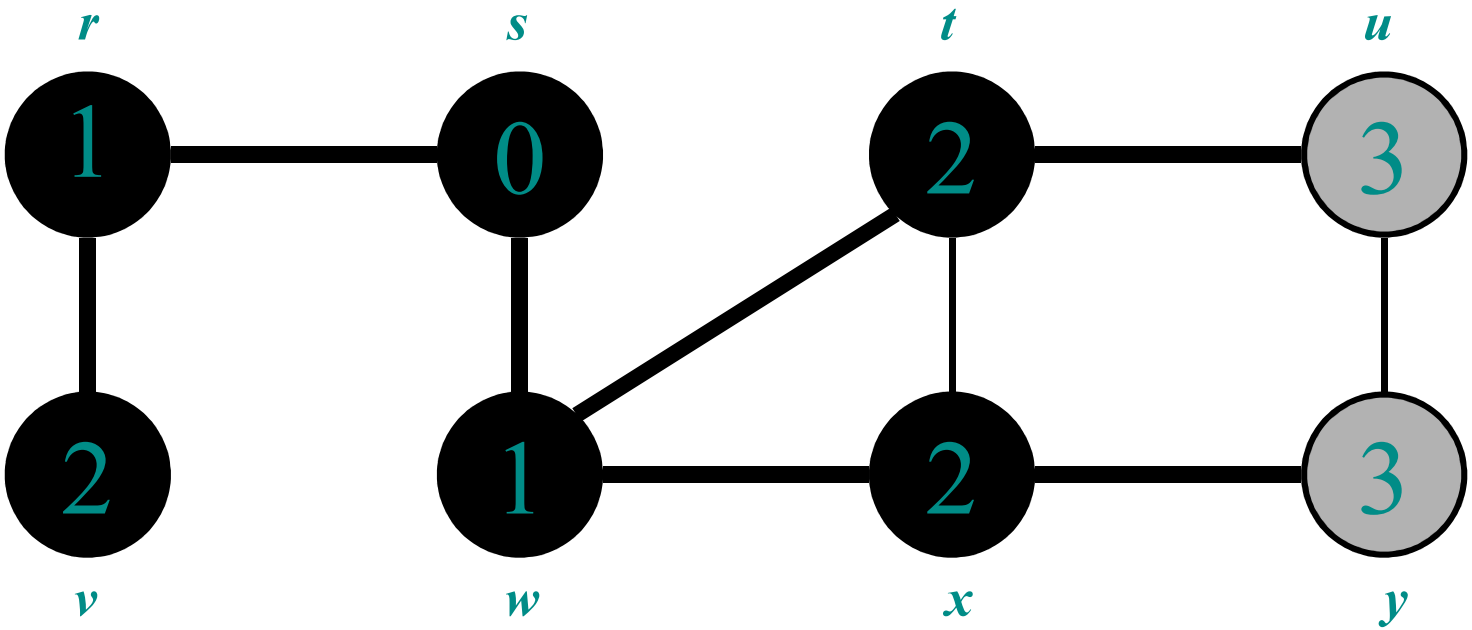
Q: | *x* | *v* | *u* |

Example



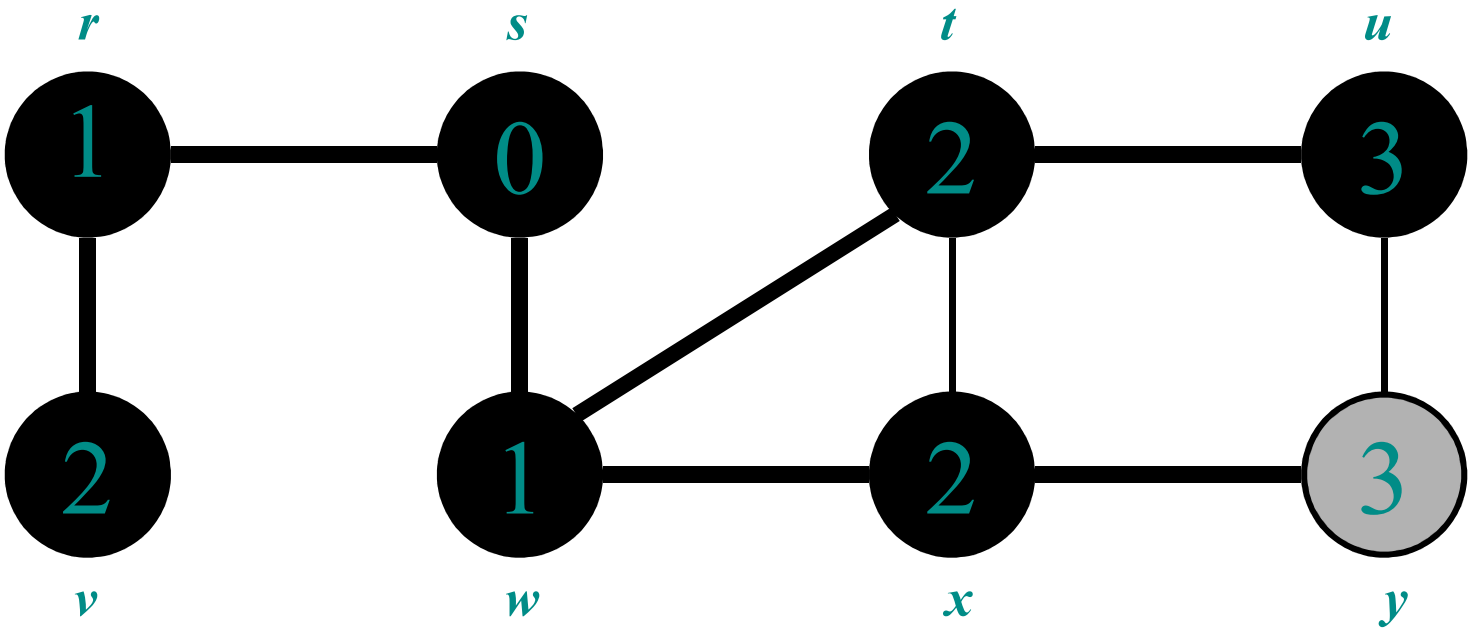
Q: | *v* | *u* | *y* |

Example



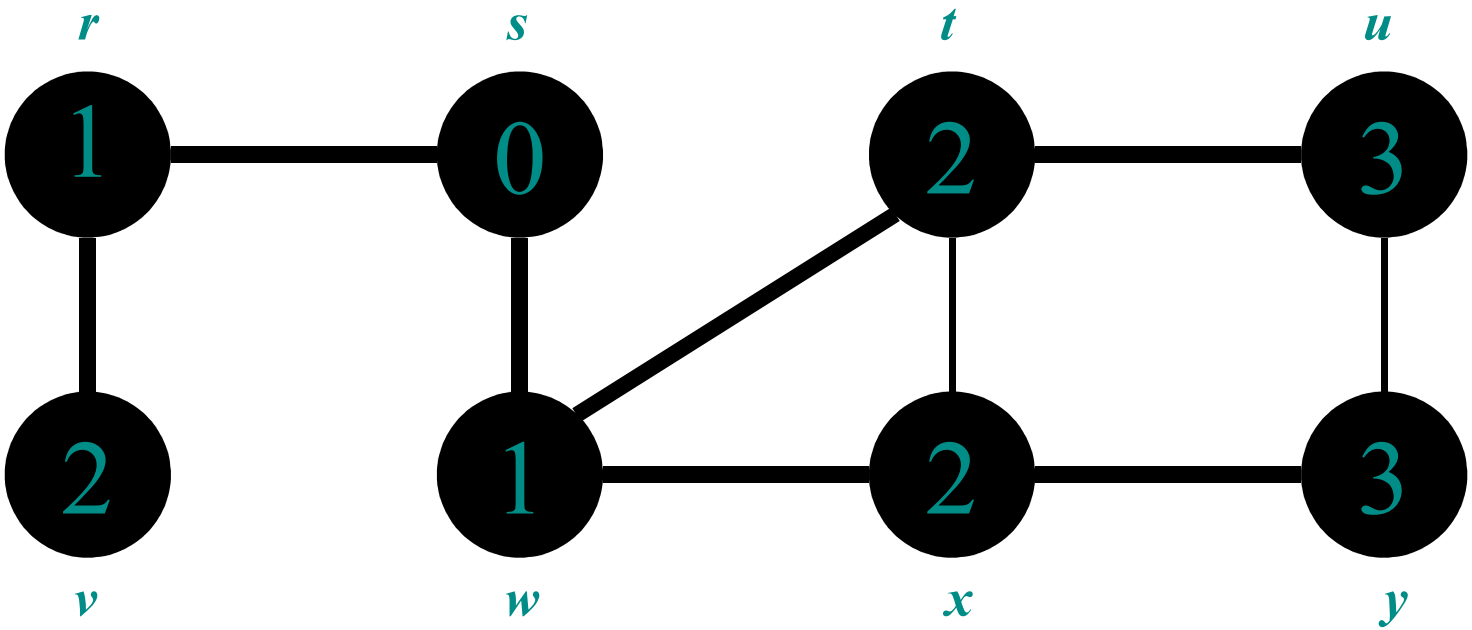
Q: | *u* | *y* |

Example



Q: *y*

Example



Q: Ø

Properties of BFS

- Proposition: Let G be an undirected graph on which a BFS traversal starting at vertex s has been performed. Then
  - The traversal visits all vertices in the connected component of s.
  - The discovery-edges form a spanning tree T, which we call the BFS tree, of the connected component of s

Properties of BFS

– For each vertex v at level i, the path of the BFS tree T between s and v has i edges, and any other path of G between s and v has at least i edges.

– If (u, v) is an edge that is not in the BFS tree, then the level numbers of u and v differ by at most one.

Properties of BFS

- Proposition: Let G be a graph with n vertices and m edges. A BFS traversal of G takes time $O(n + m)$. Also, there exist $O(n + m)$ time algorithms based on BFS for the following problems:
  - Testing whether G is connected.
  - Computing a spanning tree of G
  - Computing the connected components of G
  - Computing, for every vertex v of G, the minimum number of edges of any path between s and v.

Simple DFS: Pseudocode

```
Simple-DFS(G)
1.  for each vertex u ∈ V
2.      do color[u] ← WHITE
3.  for each vertex u ∈ V
4.      do if color[u] = WHITE
5.          then Simple-DFS-Visit(u)
```

```
Simple-DFS-Visit(u)
1.  color[u] ← BLACK
2.  for each v ∈ Adj[u]
3.      do if color[v] = WHITE
4.          then Simple-DFS-Visit(v)
```

Timed DFS: Pseudocode

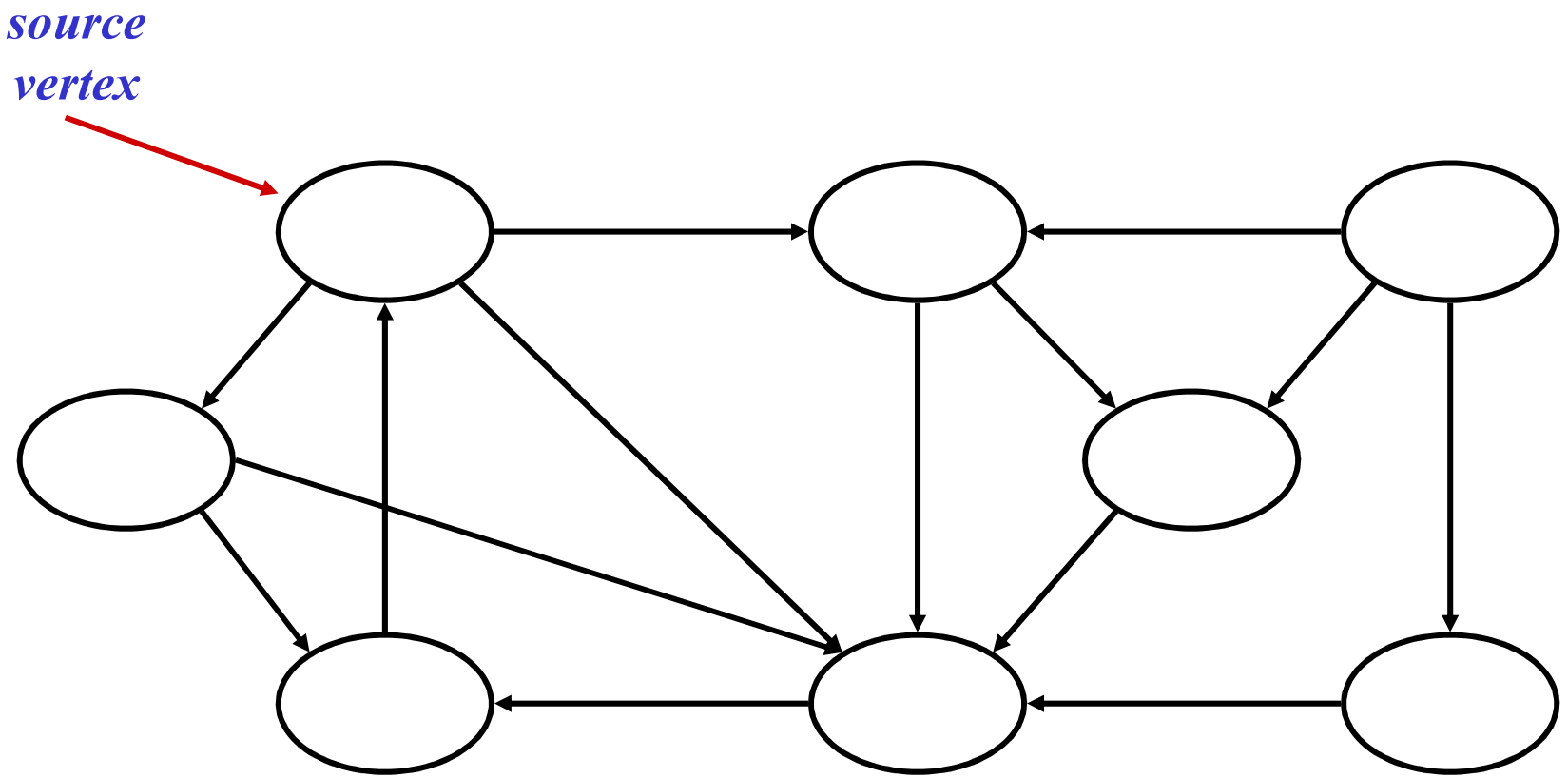
- input graph G may be directed or undirected.
- *time* is global variable used for time-stamping.

```
DFS(G)
1.  for each vertex u ∈ V
2.        do color[u] ← WHITE
3.  time ← 0
4.  for each vertex u ∈ V
5.        do if color[u] = WHITE
6.              then DFS-Visit(u)
```

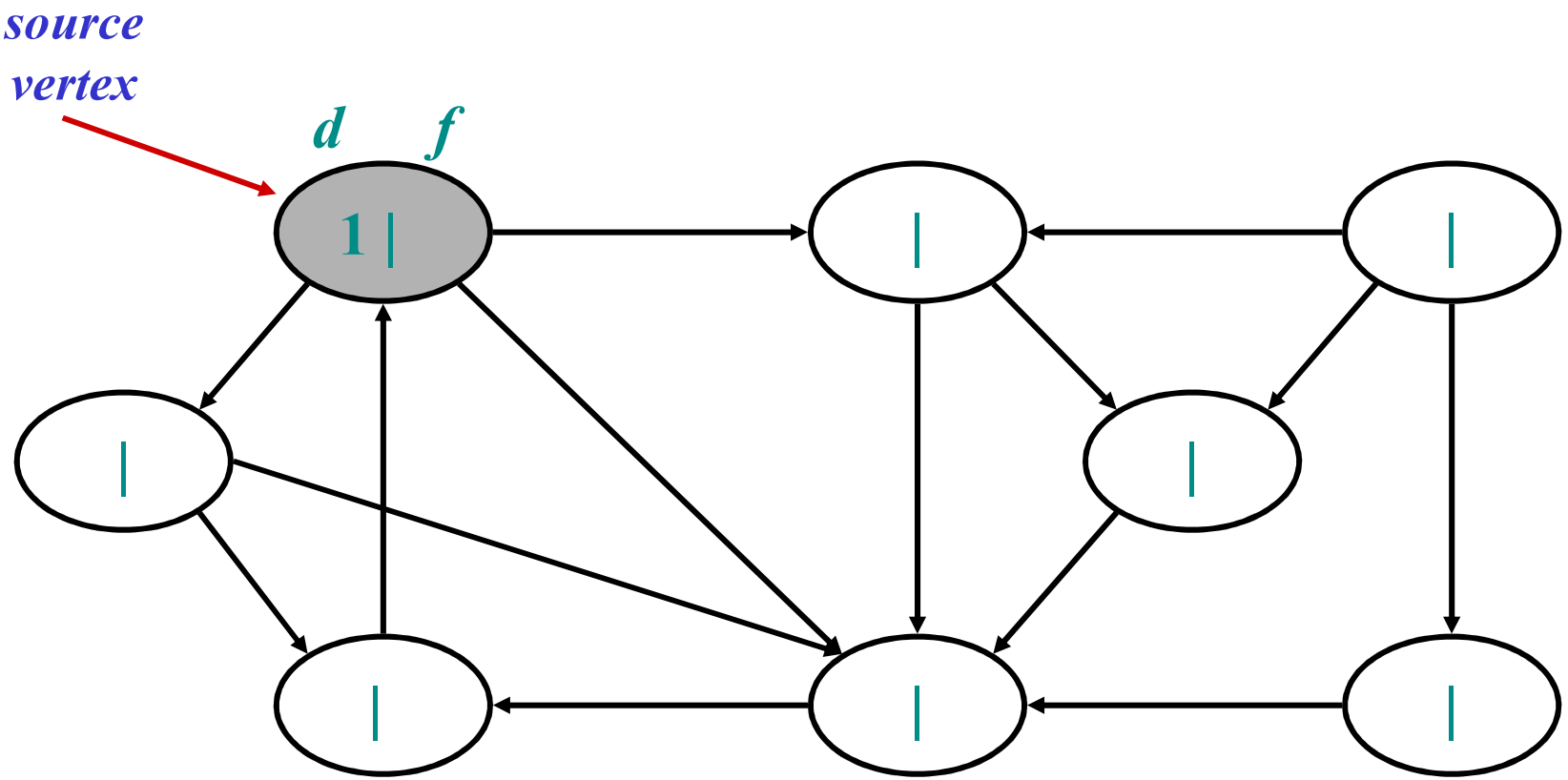
DFS-Visit(u)

1. $color[u] \leftarrow$ GRAY //White vertex u discovered.
2. $d[u] \leftarrow time$ //Mark with discovered time.
3. $time \leftarrow time + 1$ //Tick global time.
4. **for** each $v \in Adj[u]$ //Explore all edges (u, v).
5. **do if** $color[v] =$ WHITE
6. **then** DFS-Visit(v)
7. $color[u] \leftarrow$ BLACK //Blacken u; it is finished.
8. $f[u] \leftarrow time$ //Mark with finishing time.
9. $time \leftarrow time + 1$ //Tick global time.

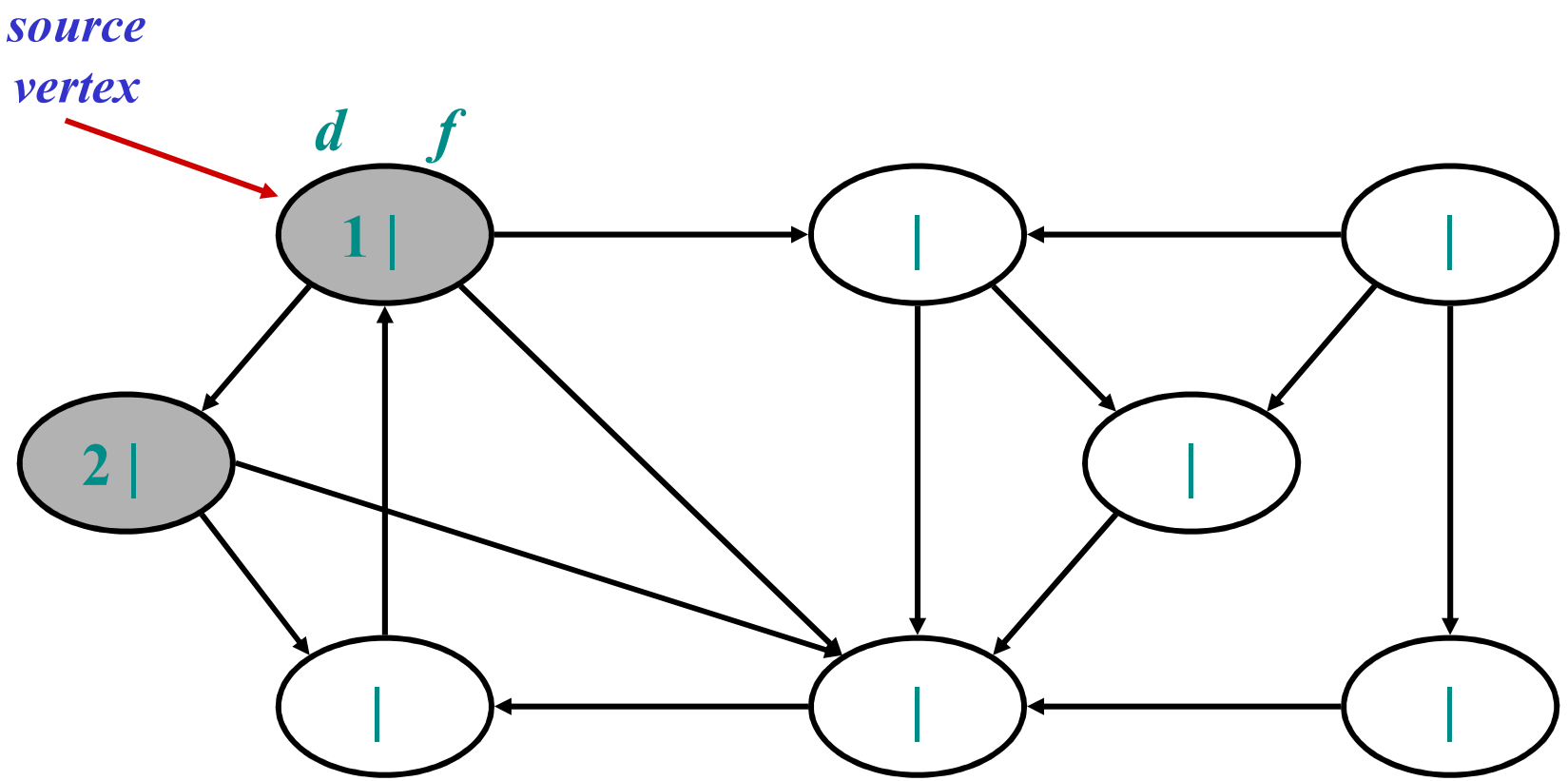
DFS Example



DFS Example



DFS Example



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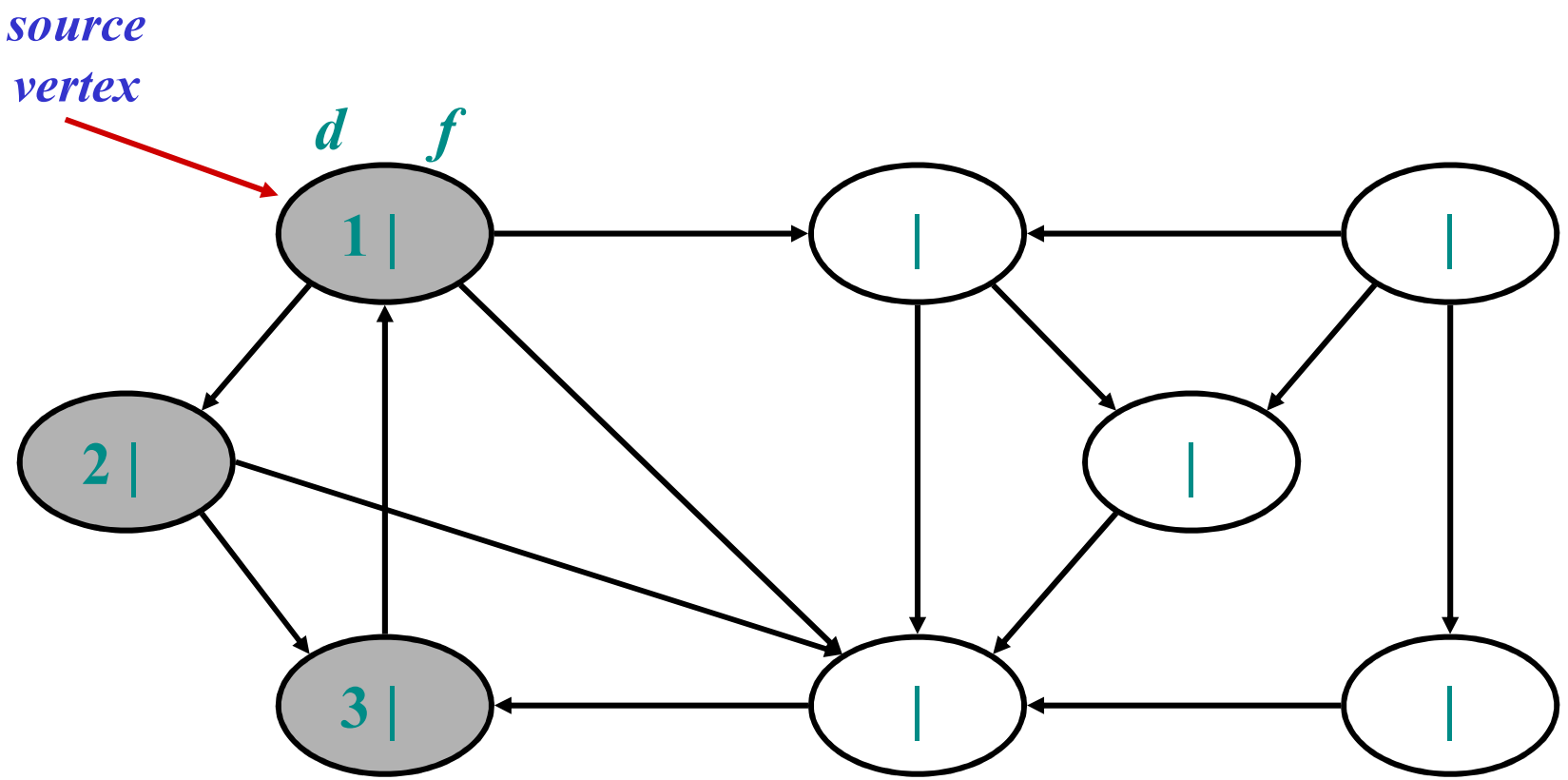

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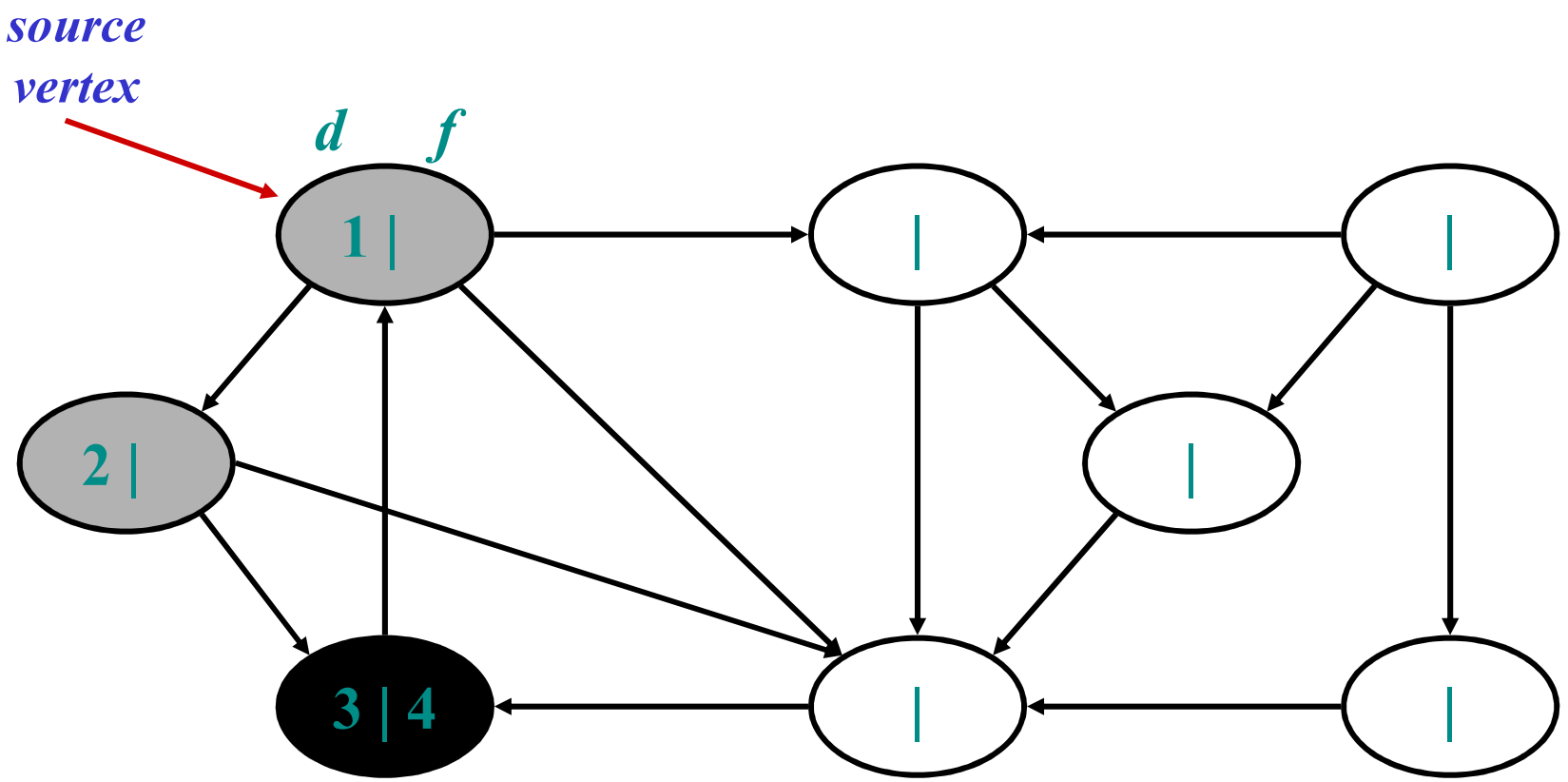

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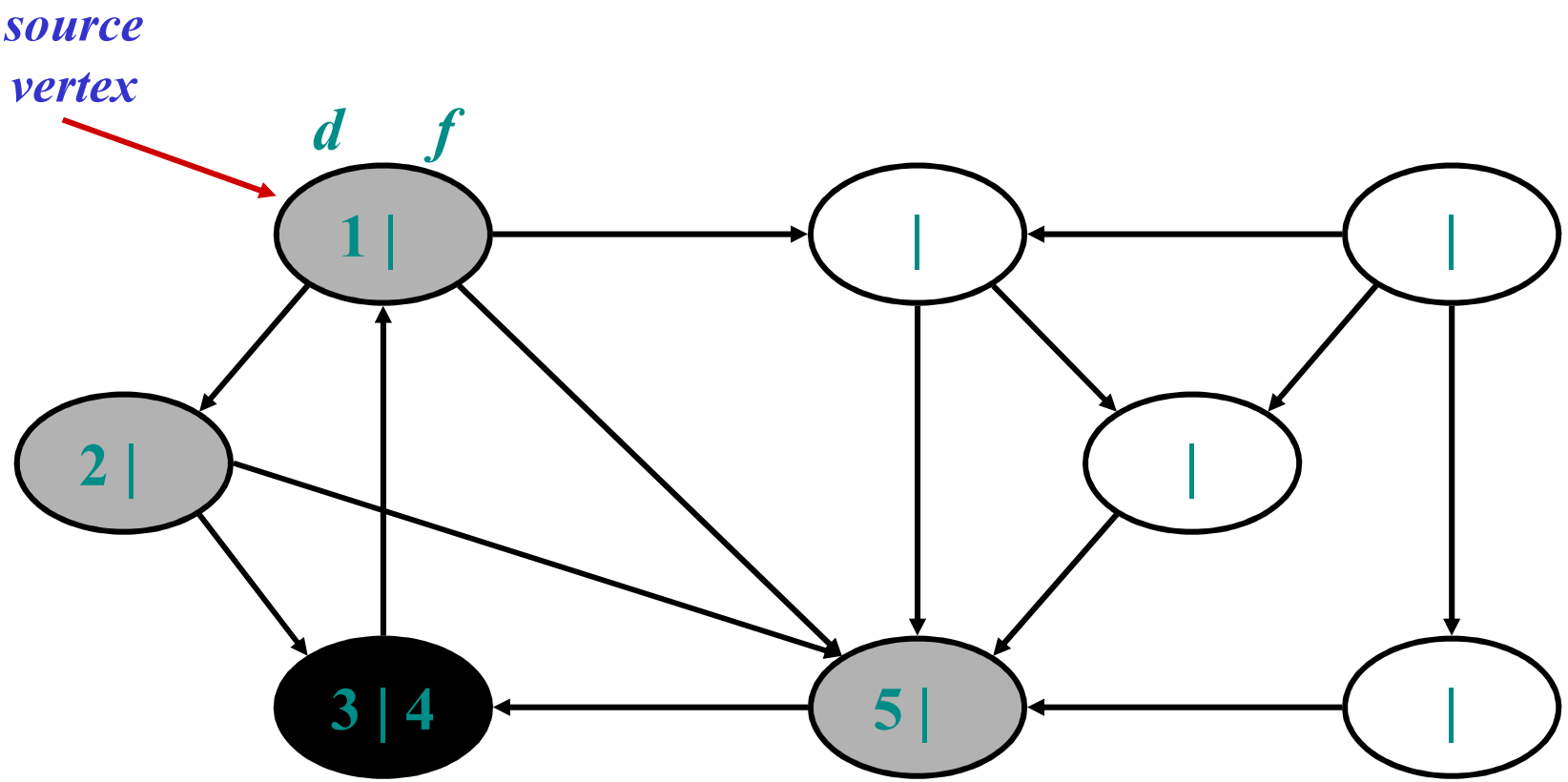

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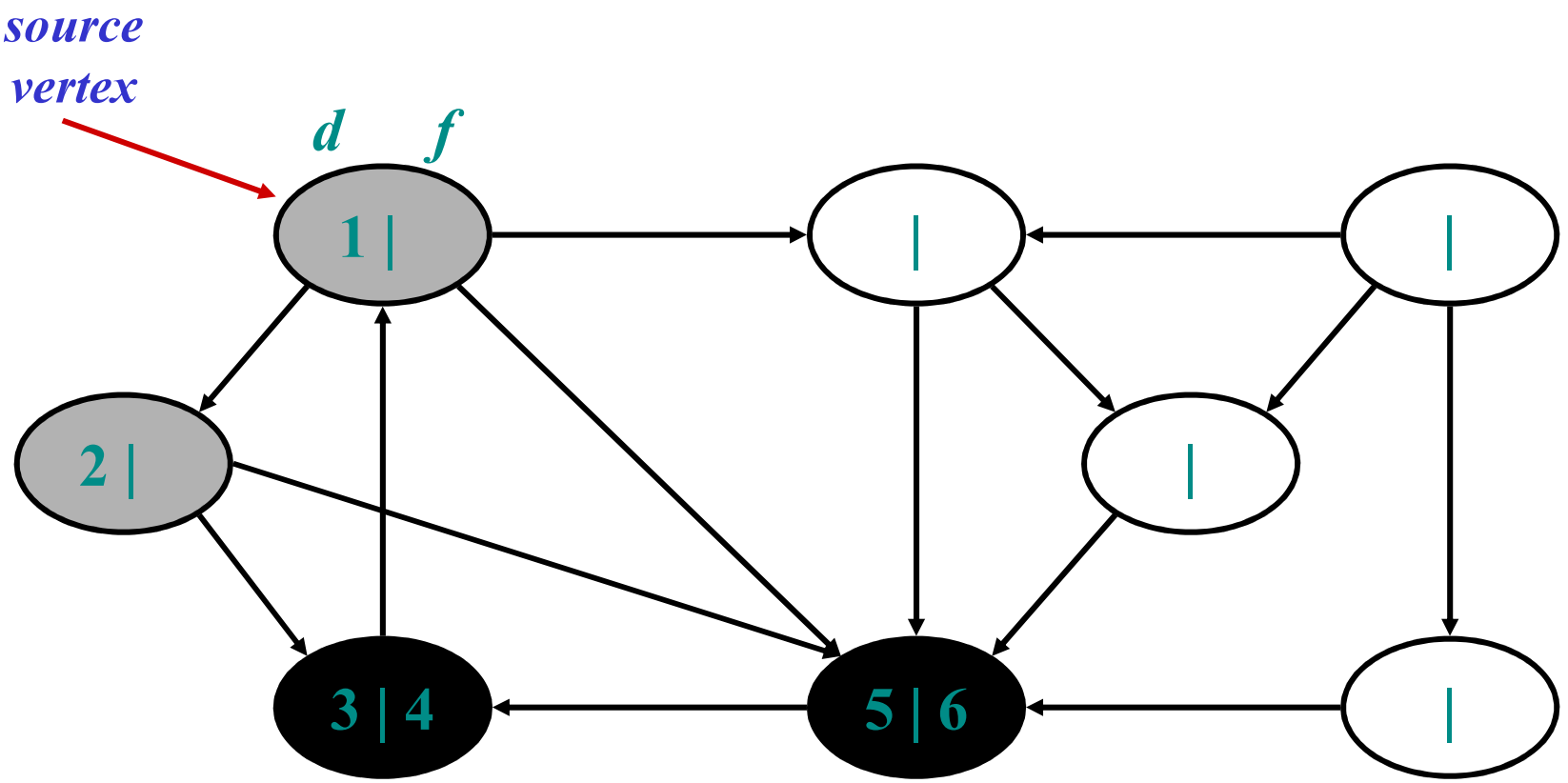

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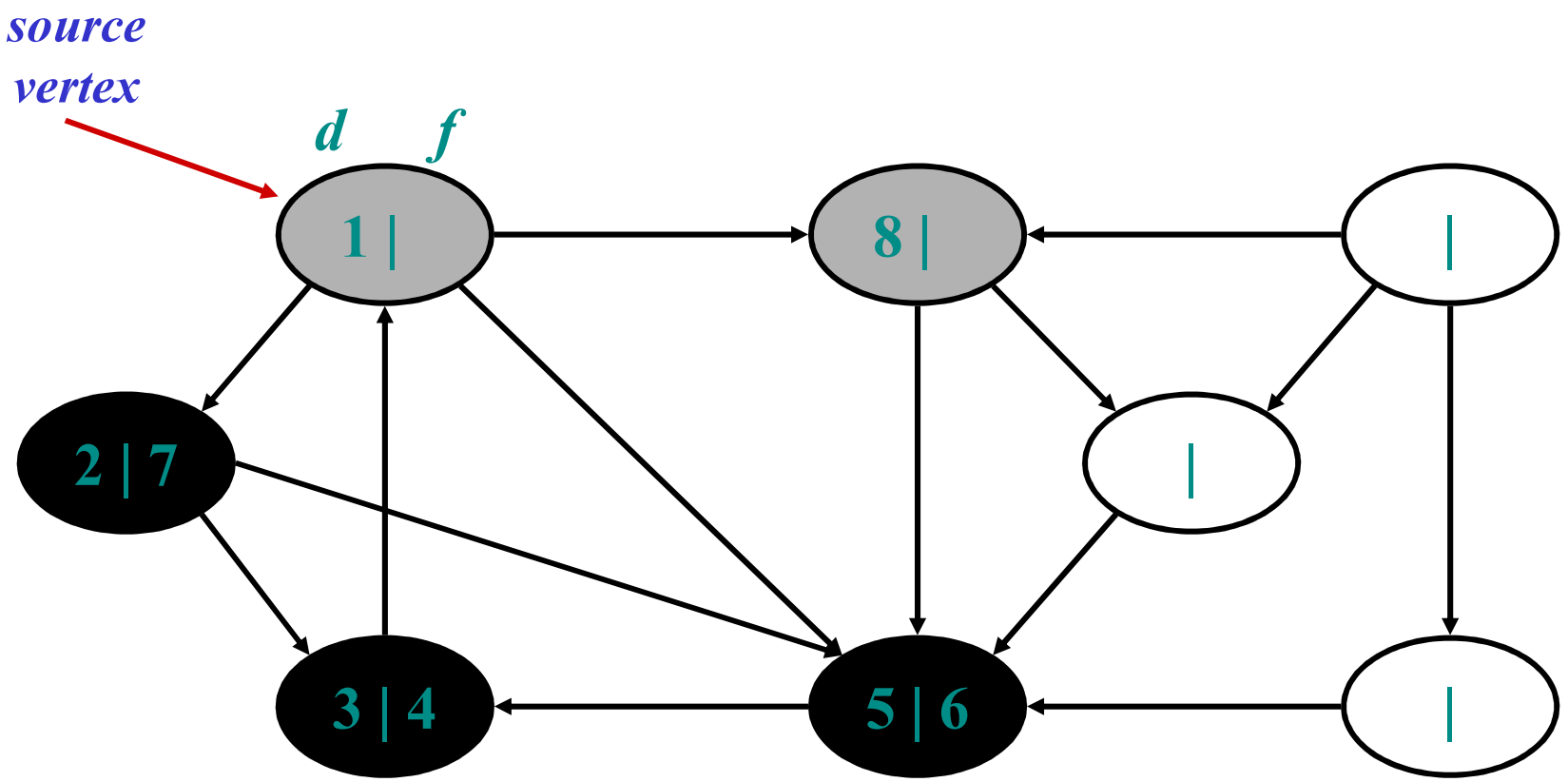

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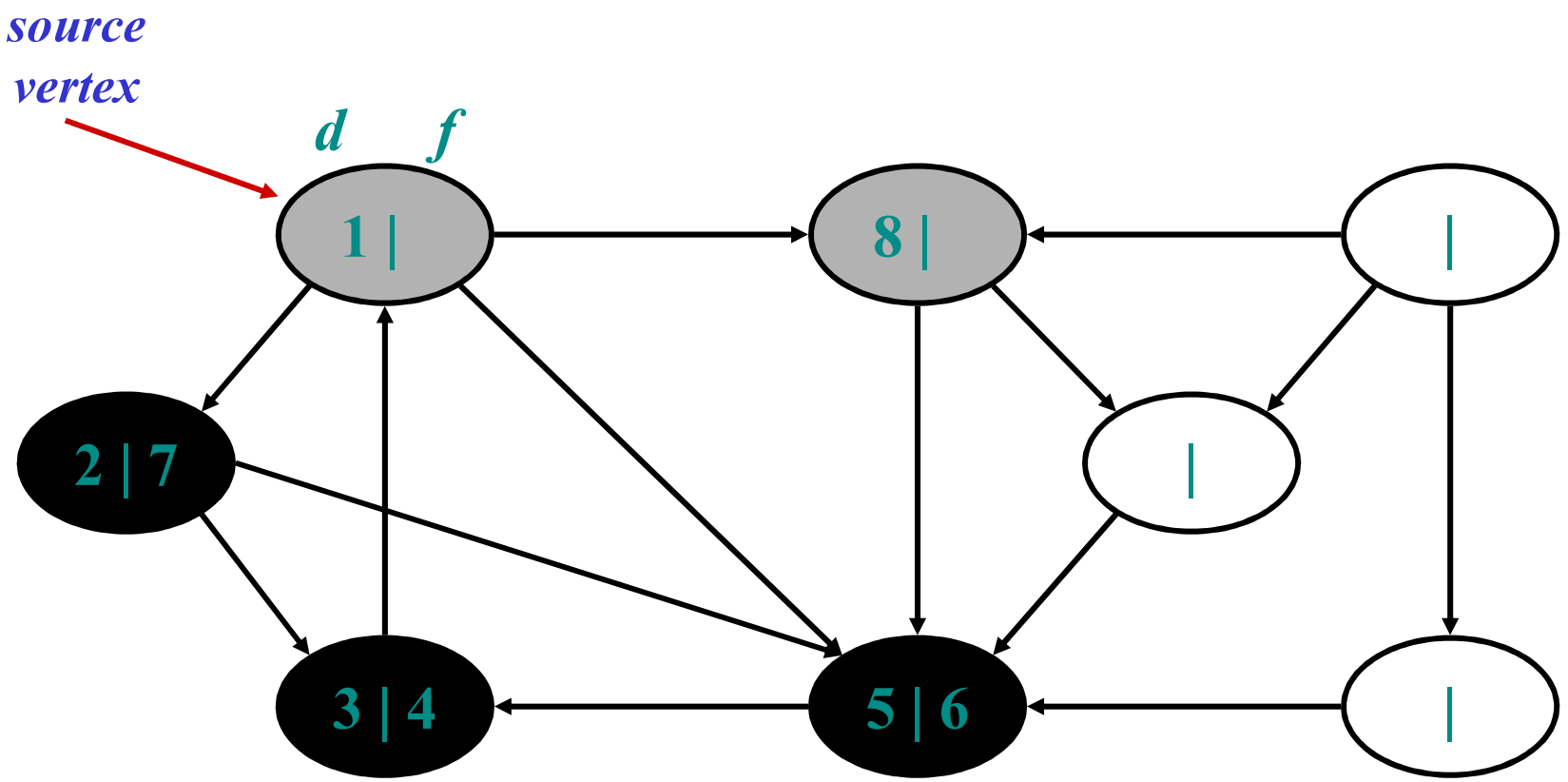

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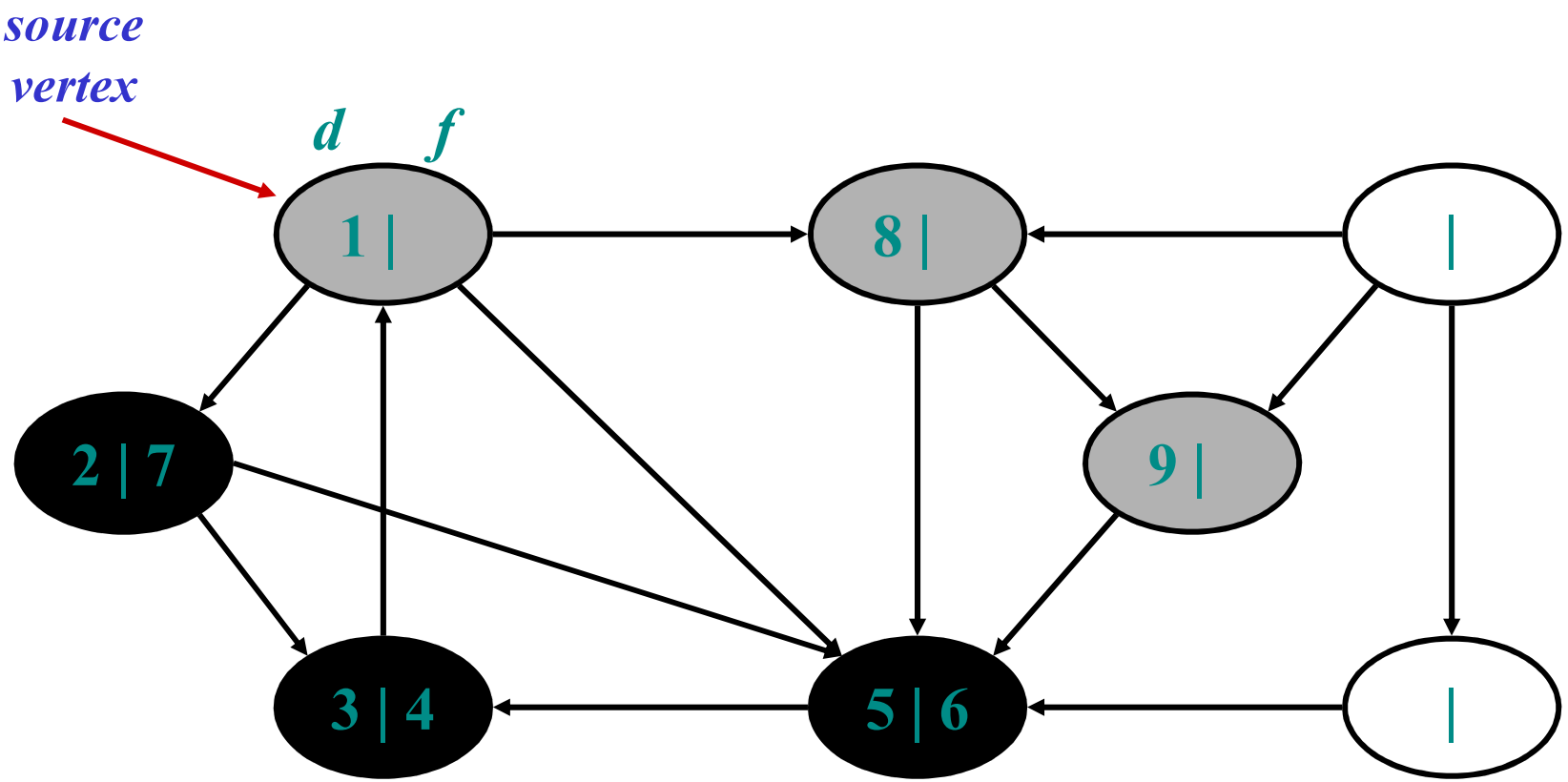
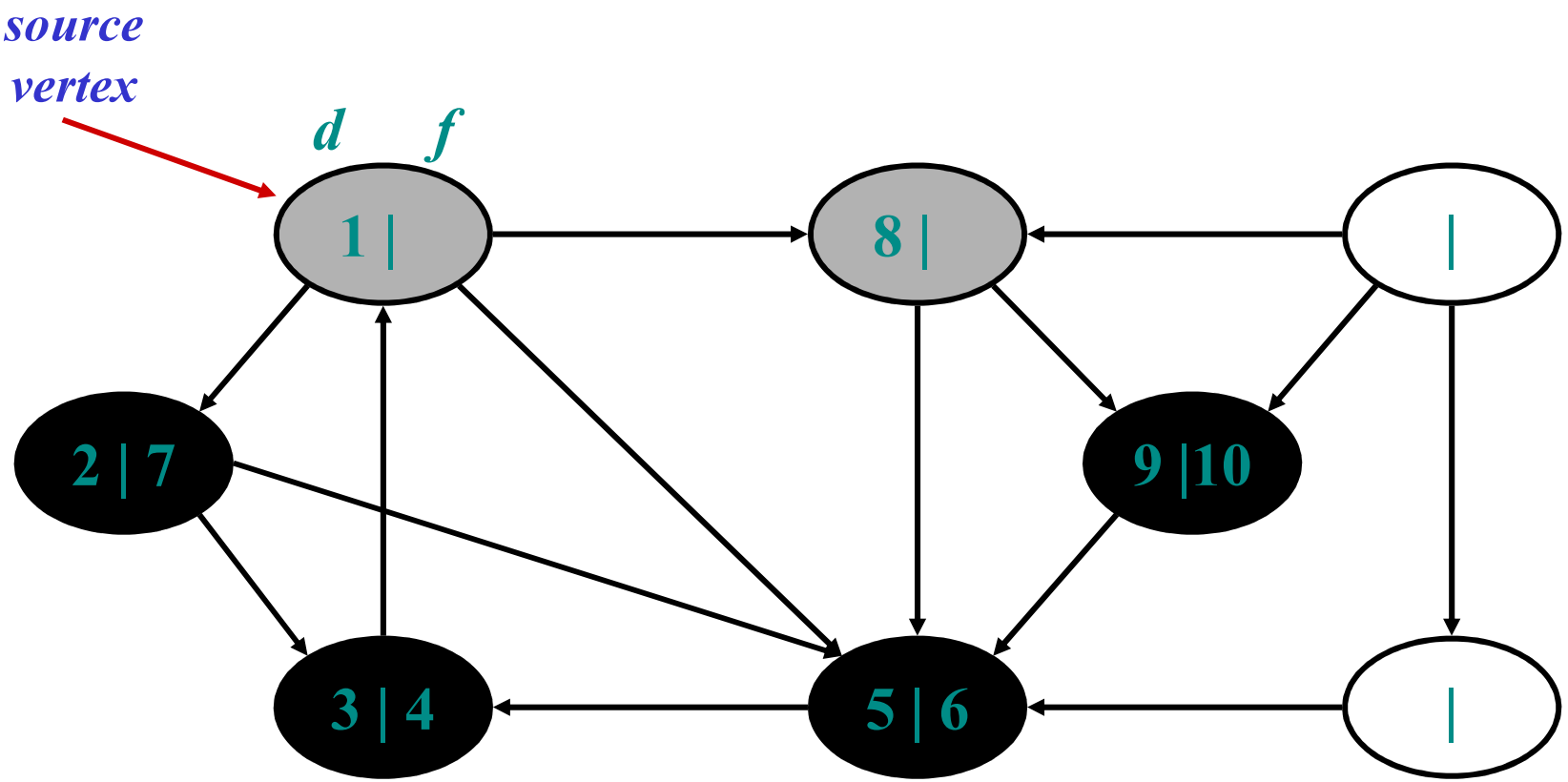


DFS Example



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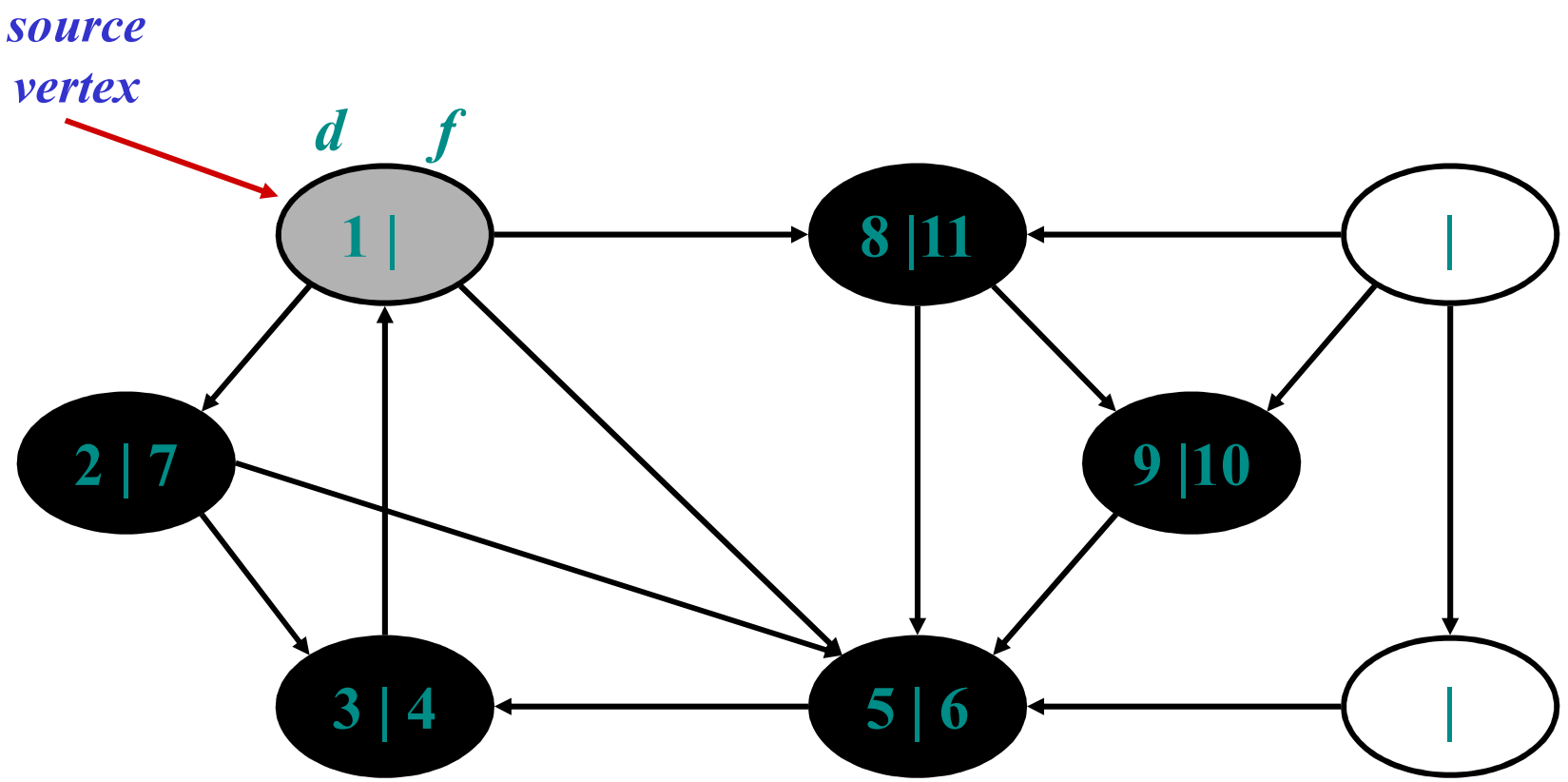

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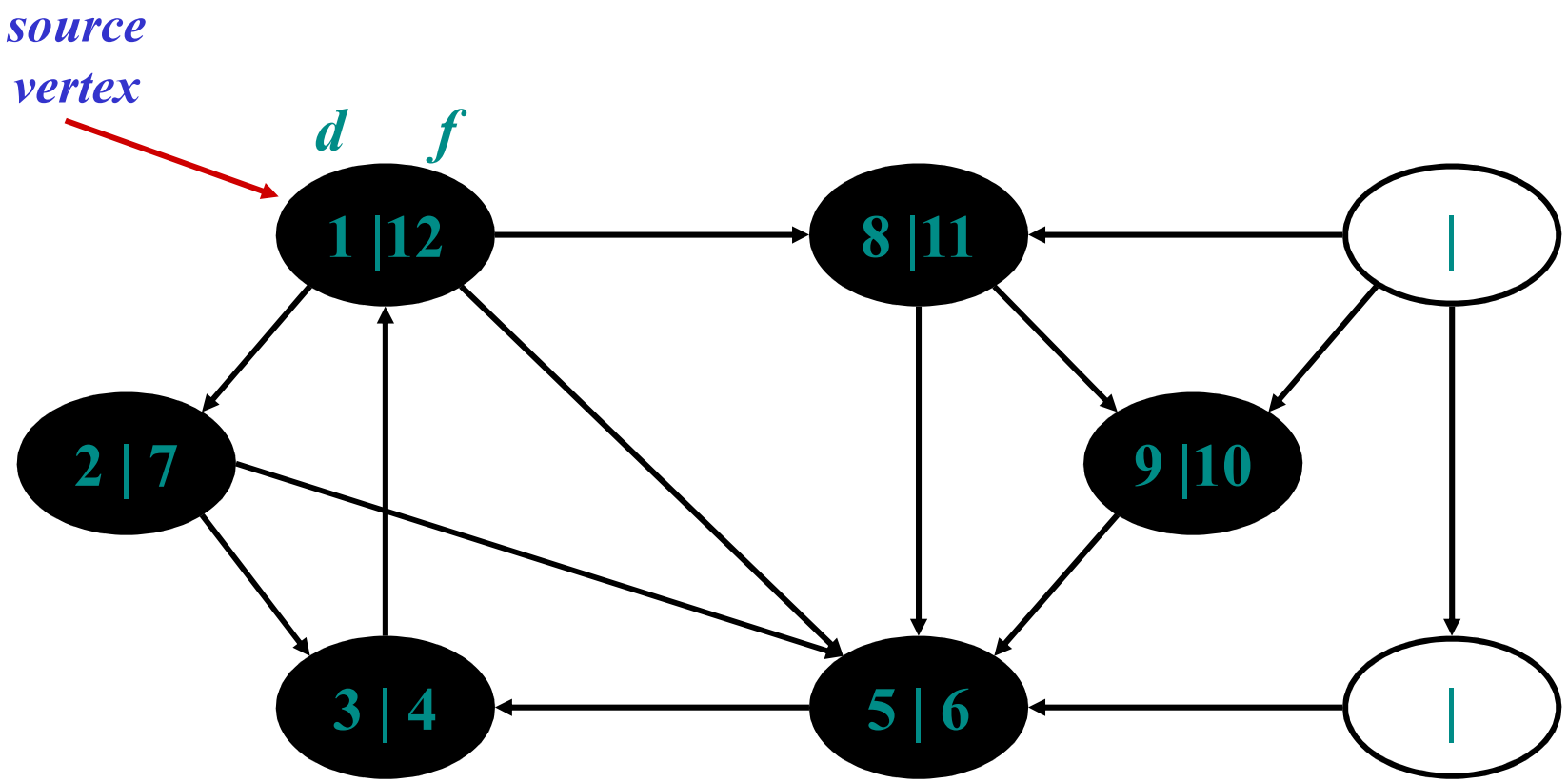

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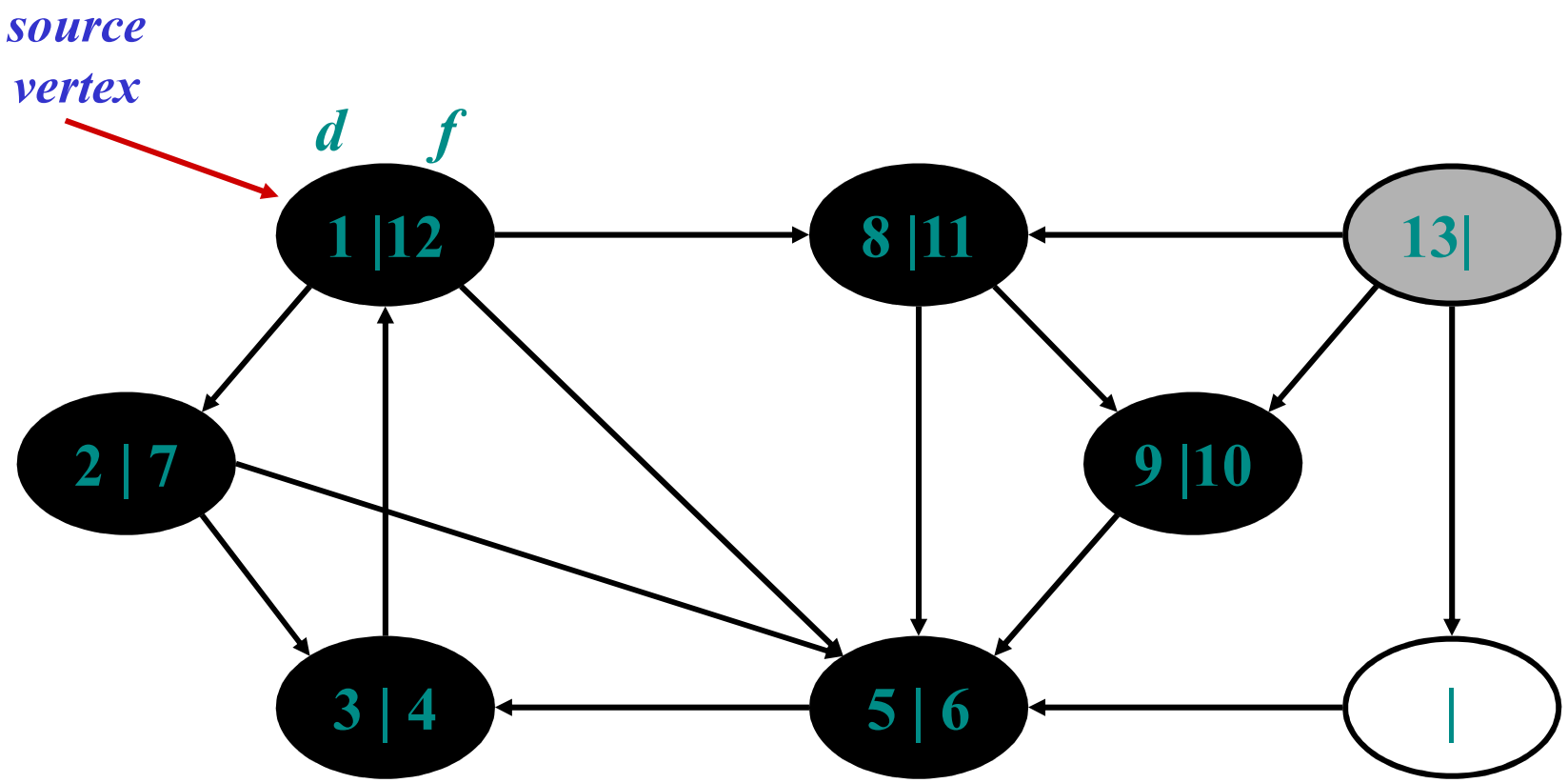

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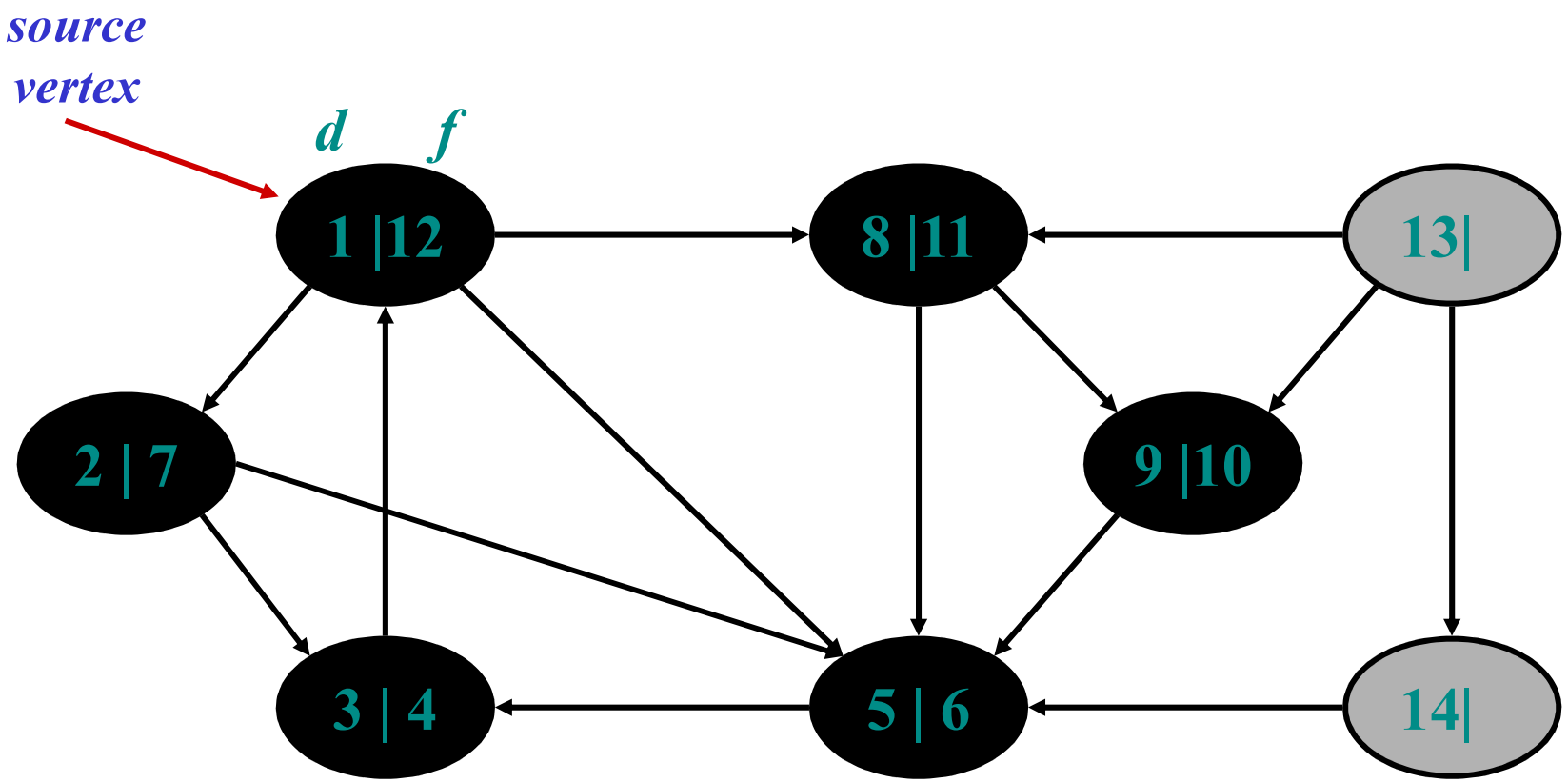
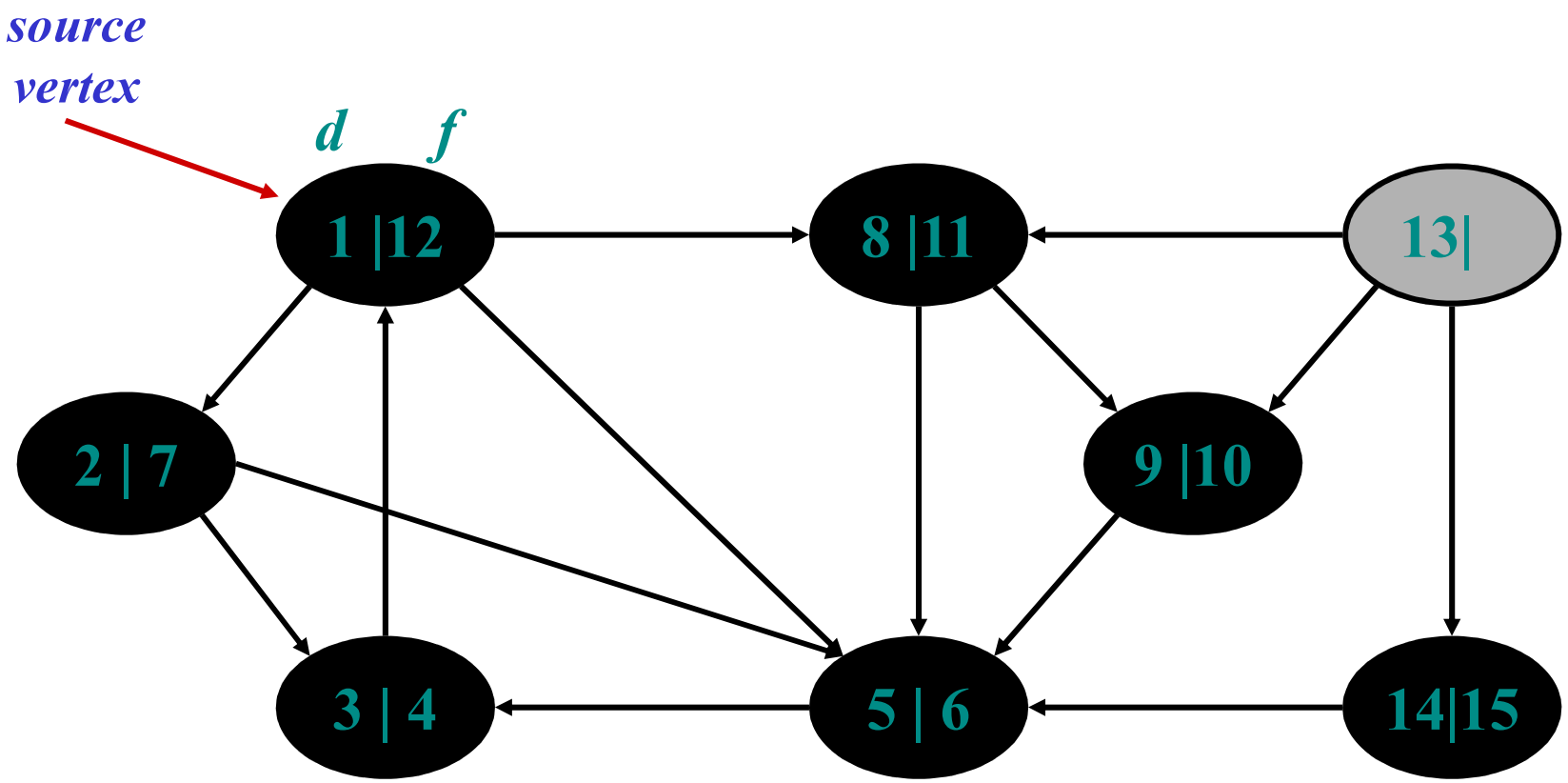
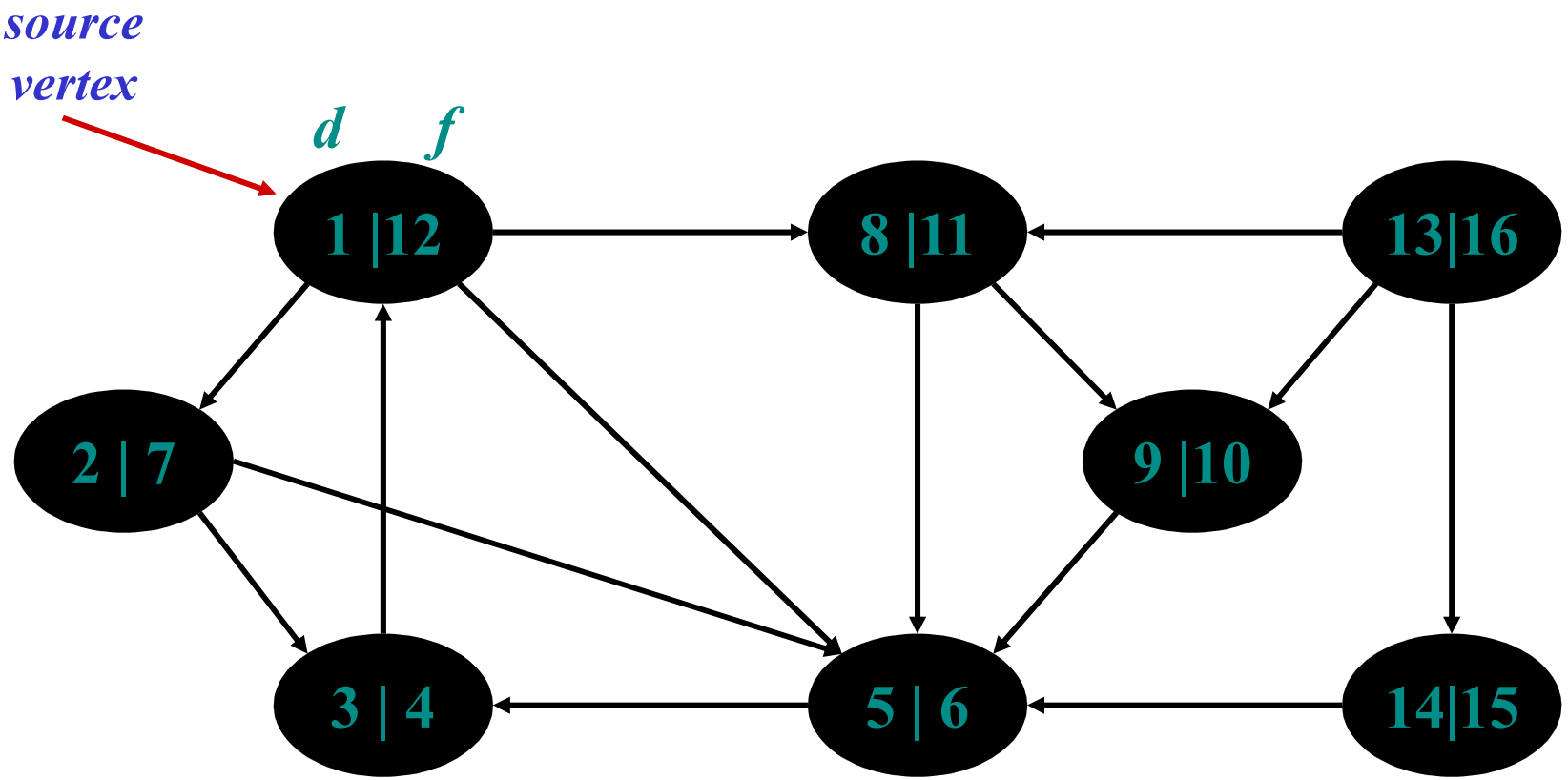


DFS Example

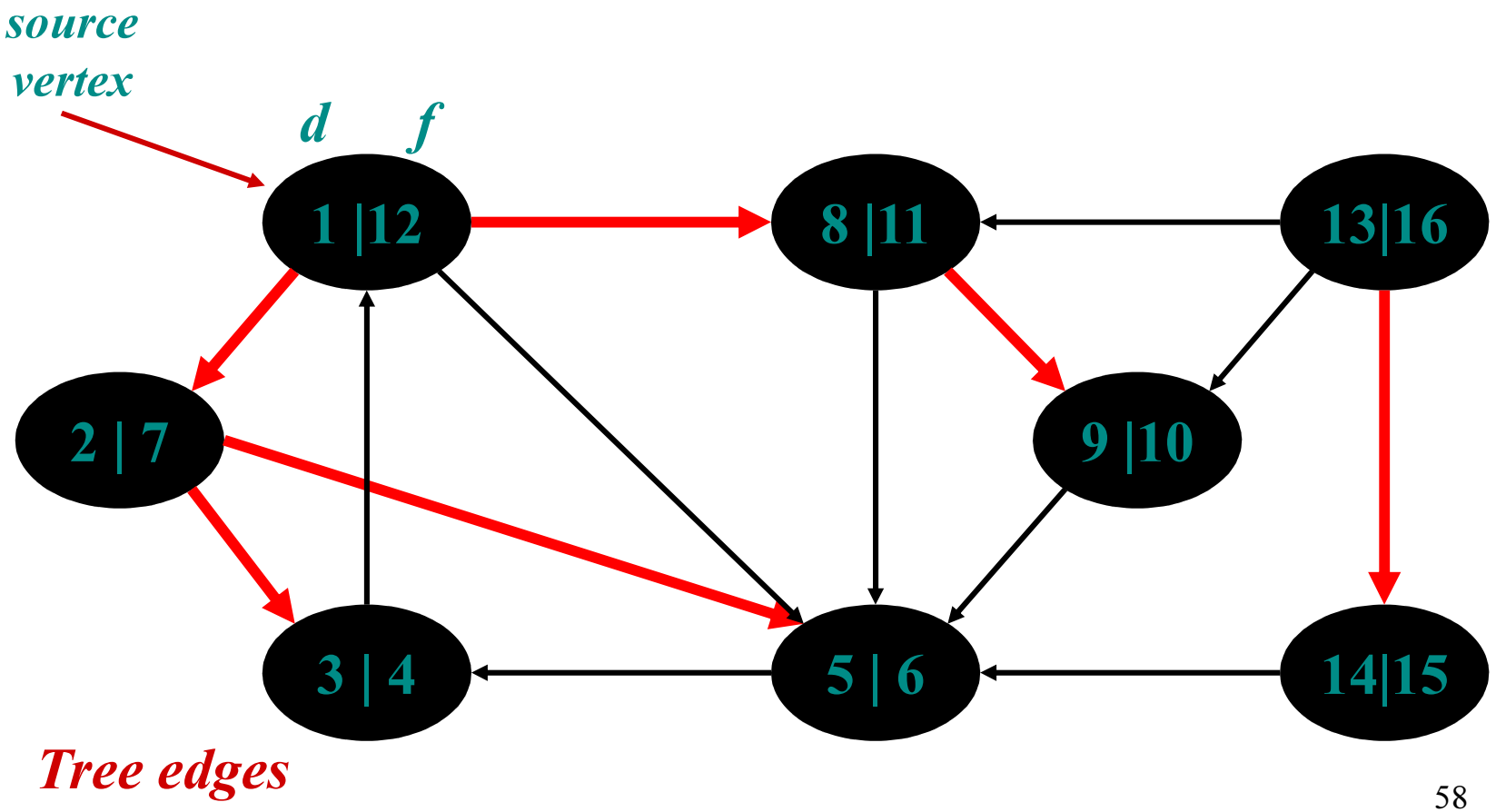


DFS Example



Properties of DFS

- Vertices traversed in search form depth-first forest of trees.



Properties of DFS

- Discovery and finishing times have parenthesis structure.
  - u is an ancestor of v if and only if $[d[u], f[u]] \supseteq [d[v], f[v]]$
  - u is a descendent of v if and only if $[d[u], f[u]] \subseteq [d[v], f[v]]$
  - u and v are not related if and only if $[d[u], f[u]]$ and $[d[v], f[v]]$ are disjoint

Negative-weight cycles

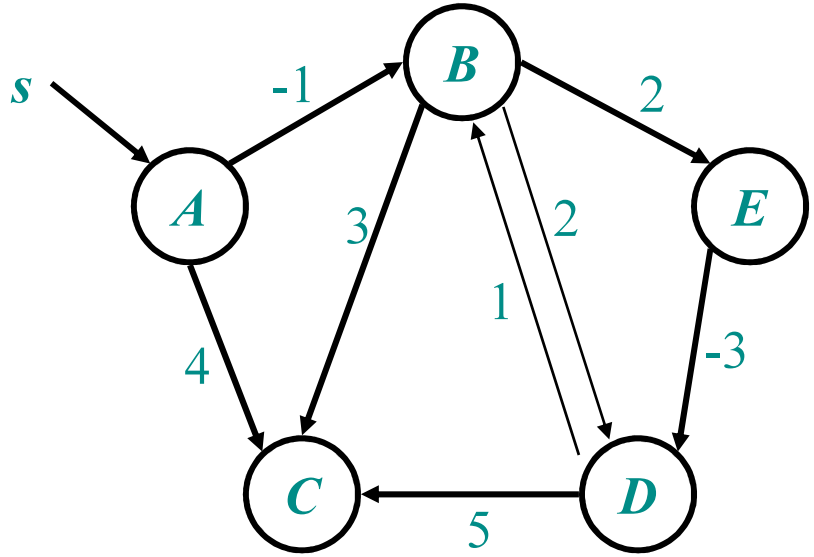
- If a graph $G = (V, E)$ contains a negative-weight cycle, then some shortest paths may not exist.
- Bellman-Ford algorithm: Finds all shortest-path lengths from a source $s \in V$ to all $v \in V$ or determines that a negative-weight cycle exists.

Bellman-Ford Algorithm

- $d[s] \leftarrow 0$
- **for** each $v \in V - \{s\}$
  do $d[v] \leftarrow \infty$
- **for** each $i \leftarrow 1$ to $|V| - 1$
  do for each edge $(u, v) \in E$
  do if $d[v] > d[u] + w(u, v)$
  then $d[v] \leftarrow d[u] + w(u, v)$
- **for** each edge $(u, v) \in E$
  do if $d[v] > d[u] + w(u, v)$
  then report that a negative-weight cycle exists

Time = $O(VE)$

Example of Bellman-Ford



work on board

Correctness

- **Theorem** If $G = (V, E)$ contains no negative-weight cycles, then after the Bellman-Ford algorithm executes, $d[v] = \delta(s, v)$ for all $v \in V$.

- **Corollary** If a value $d[v]$ fails to converge after $|V| - 1$ passes, there exists a negative-weight cycle in G reachable from s.